ON BINARY SUM MATRICES

SASHA RAVSKY

We are going to describe all $m \times n$ sum matrices modulo 2. This problem is well known and has different names with *Lights Out* used more often then others. For the definitions, notations and proofs of the following results I recommend the following list of links in the separate file. Most of definitions are from the paper at http://www.unf.edu/~wkloster/termpaper.pdf. All calculations are considered over the field $\mathbb{Z}_2$ of integers modulo 2.

For the fixed m and n as $R \subset \mathbb{Z}_2^{mn}$ denote the linear space of all sum matrices (it is called *a null space*). Our aim is to describe the basis of the space R and his dimension $R(m,n)$. For instance, for the rectangle $R(5,5) = 2$ and the R has the basis vectors

$$\begin{array}{ccc} 10101 & & 11011 \\ 10101 & & 00000 \\ 00000 & \textit{and} & 11011 \\ 10101 & & 00000 \\ 10101 & & 11011 \end{array}$$

This was done in the cited papers (most from the Klostermeyer's site) by the means of Fibonacci polynomials over $\mathbb{Z}_2$. The Fibonacci polynomials are defined as $f_0(x) = 0$, $f_1(x) = 1$, $f_2(x) = x$ and $f_i(x) = x f_{i-1}(x) + f_{i-2}(x)$.

The following proposition gives the full solution of the problem and the bases of R can be obtained using the theory of Fibonacci polynomials over $\mathbb{Z}_2$ (see the links).

Proposition 1. $R(m,n) = \deg \gcd(f_{m+1}(x), f_{n+1}(1+x))$

Proposition 2. *If $k \geq 3$ and $n = 2^k$ or $n = 2^k - 2$ then $R(n,n) > 0$.*

Proposition 3. *If $n \neq 4$ then $R(n,n) < n$.*

There are following results for small m.
$R(1,n) = 0$ iff $3 \nmid (n+1)$
$R(2,n) = 0$ iff $2 \nmid (n+1)$
$R(3,n) = 0$ iff $3 \nmid (n+1)$
$R(4,n) = 0$ iff $5 \nmid (n+1)$
$R(5,n) = 0$ iff $2 \nmid (n+1)$ and $3 \nmid (n+1)$
$R(6,n) = 0$ iff $9 \nmid (n+1)$
$R(24,n) = 0$ iff $5 \nmid (n+1)$.

For finding values of m, n with $R(m,n) = 0$ we need only consider even values of n, since the characterization for odd n may be deducted directly using the next proposition.

Proposition 4. *Suppose that $n > 2$ is an odd positive integer such that $n+1 = 2^i(q+1)$ where q is even and i is a positive integer. Then $R(m,n) = 0$ iff $R(m,q) = 0$ and $m+1$ is not a multiple of 3.*

Proposition 5. *If $m = 5 \cdot 2^i - 1$, where i is any positive integer, and if $n = 3m + 2$ then $R(m,n) \geq m$.*

Proposition 6. *If Fibonacci polynomial f_{m+1} is the square of irreducible polynomial and if $n+1$ is Fibonacci index of $f_{m+1}(1+x)$ then $R(m,n) \geq m$.*

The list of all $n \leq 10,000$ such that $R(n,n) = 0$ is at *http://www.unf.edu/ wkloster/fibonacci/solvable.*

The following *conjectures* was suggested by Rafael Losada and his Java Solver of Lights Out from http://www.anarkasis.com/rafa/homeuk.htm

If $3|(n+1)$ then $R(1,n) = 1$ and R has the basis in the form

$$110110110\ldots$$

Suppose that $2|(n+1)$. Then $R(2,n) \geq 1$ and R contains the vector

$$\begin{array}{c} 10101\ldots \\ 10101\ldots \end{array}$$

Moreover, if $4|(n+1)$ then $R(2,n) = 2$ and R contains the vector

$$\begin{array}{c} 01000010\ldots \\ 11101110\ldots \end{array}$$

At last, if $2|(n+1)$ and $4 \not| (n+1)$ then $R(2,n) = 1$.

If $3|(m+1)$ and $2|(n+1)$ then $R(m,n) \geq 1$ and R contains the vector

$$\begin{array}{c} 10101\ldots \\ 10101\ldots \\ 00000\ldots \\ 10101\ldots \\ 10101\ldots \end{array}$$

I expect that the general problem about sum matrices can be solved with similar approach.

E-mail address: `oravsky@mail.ru`

Department of Mathematics, Ivan Franko Lviv National University, Universytetska 1, Lviv, 79000, Ukraine