Theorem 1. *If $F(n)$ is the number of Friedman numbers in the range $[1, n]$, then* $\lim_{n\to\infty} F(n)/n = 1$.

In order to prove this claim, we begin by defining a class of integers that will be used throughout the proof.

Definition: Let an integer, x, be called **basic** if it is greater than 1 and is not a non-trivial integer power.

Also,

Define the **span** of a positive integer, x, to be the set of positive integers that can be produced from its digits by use of the four basic arithmetic operations, exponentiation and digit concatenation.

As an example, note that the Friedman numbers can be defined as the set of integers, x, for which $x \in span(x)$.

We will additionally use the notation $[s]^k$ to mean "the number that is the concatenation of k copies of the digits of the number s".

Lemma 1.1. *If $x_1, \ldots, x_t$ are t basic integers (not necessarily distinct) and x is the integer produced by the digit concatenation of $x_1, \ldots, x_t$, then $|span(x)|$ is greater or equal to the number of distinct order permutations of $x_1, \ldots, x_t$.*

Proof. Let $y_1, \ldots, y_t$ be an order permutation on $x_1, \ldots, x_t$, and consider the number $y = y_1^{y_2^{y_3\cdots}}$. Because all y_i are basic integers, it is possible to deduce all of $y_1, \ldots, y_t$ from y, meaning that any two distinct order permutations necessarily lead to different values of y. □

There are 5 basic integers that are 1 digit long, 82 basic integers that are 2 digits long and 872 basic integers that are 3 digits long. Let us define the constant s to be the digit concatenation of 13 copies of each of these basic integers. The digit length of s, $L(s)$, is $2785 * 13 = 36205$, and due to the lemma we know that $|span(s)| \geq (959 * 13)!/13!^{959}$, which is a value we will name g.

Lemma 1.2. *Let k be a positive integer that is congruent to $-6 \pmod{s-3}$, let a and b be non-negative integers, let $m \in span([s]^k)$ and let $r = \dfrac{k+6}{s-3}$. The value*

$$N(k, a, b, m) = \left(a * \left(10^{L(s)}\right)^{rs} + \frac{\left(10^{L(s)}\right)^{rs} - 1}{10^{L(s)} - 1} s\right) * \left(10^{L(s)}\right)^{(rs)m} + b$$

is a Friedman number, if $L(b) \leq L(s)rsm$, where $L(b)$ is the digit length of b.

Proof. If $L(b) = L(s)rsm$, then the digits of $N(k, a, b, m)$ are the digits of a, the digits of b, and rs copies of s. If $L(b) < L(s)rsm$ then $N(k, a, b, m)$ additionally contains some zeros between the repetitions of s and the beginning of b. In calculating $N(k, a, b, m)$ from its digits, these zeros can be taken care of by adding them to the final result.

It can easily be shown that the constants 1, 10, s, $L(s)$, $10^{L(s)}$ and $10^{L(s)} - 1$ are all in the span of s. (This is trivial to do, because each of the ten digits is represented in s more than 2000 times.) m, per definition, can be constructed from k copies of s, and rs, which is the only remaining part of the equation that defines $N(k, a, b, m)$, can be represented as the summation $\underbrace{s + \ldots + s}_{r\ times}$. This means that $N(k, a, b, m)$ can be fully calculated by the use of the digits of a, of b, of the zeros before b if such exist, and by $6 + 3r + k$ copies of s. Because, by definition, $6+3r+k = rs$, $N(k, a, b, m)$ is a Friedman number. □

We will now show that $F(n)/n \rightarrow 1$ even if the function F counts only Friedman numbers of the type $N(k, a, b, m)$.

Let m_{max} be the largest element in $span([s]^k)$. Let $w \geq L(s)rs(m_{max}+1)$. Consider $n = 10^w$. In order to simplify the discussion, let us assume that x is an integer random variable, uniformly distributed in $[1, n]$, and we wish to evaluate the probability that there exist k, a, b and m s.t. $x = N(k, a, b, m)$.

If we keep k and m as constants, but allow a and b to take any value, then the probability of x belonging to the set $\{N(k, a, b, m)|\forall a, \forall b\}$ is $10^{-rsL(s)}$. This is because the set contains all numbers for which a specific set of $rsL(s)$ digits matches a particular value. It has no other restrictions.

The probability that x is not $N(k, a, b, m)$ with any a, b and m but for a specific k is $\left(1 - 10^{-rsL(s)}\right)^{\left|span\left([s]^k\right)\right|}$. This is so because the sets for each value of m give independent probabilities, due to the fact that there is no overlap in the sets of digits that are restricted for any two values of m.

Our aim is to show that this value drops to zero, which is true if $\frac{\left|span\left([s]^k\right)\right|}{10^{rsL(s)}}$ tends to infinity as k rises to infinity.

The value of $\left|span\left([s]^k\right)\right|$ can be bounded from below by g^k. This is true because the total number of distinct order permutations in $13k$ copies of the basic integers of up to 3 digits cannot be less than the number of distinct order permutations that can be reached by permuting the order strictly within each set of 13 copies.

The value of $10^{rsL(s)}$ equals $10^{(k+6)L(s)\frac{s}{s-3}} = \left(10^{L(s)\frac{s}{s-3}}\right)^k * 10^{6L(s)\frac{s}{s-3}}$, where the right multiplicant is a constant.

The value of $\frac{\left|span\left([s]^k\right)\right|}{10^{rsL(s)}}$, therefore, tends to infinity with k if $g > 10^{L(s)\frac{s}{s-3}}$. Either direct calculation or use of Stirling's formula can be used to verify that this is, indeed, so. g is a value with 36258 digits, whereas $10^{L(s)\frac{s}{s-3}}$ has only $L(s) + 1 = 36206$ digits.

This proves that as n increases, the probability of an integer r.v. uniformly sampled in $[1, n]$ to be a Friedman number approaches 1, which is what we wanted to prove.

Q.E.D.