ON SUBSETS AVOIDING LARGE RECTANGLES

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Let $I = [0; 1]$ be the unit segment, $Q = I^2$ be the unit square, $n \geq 0$ and $z = (z_1, \dots, z_n) \in Q^n$. As $T(z)$ we define the largest possible area of the unrotated rectangle $R \subset Q$ (which sides are parallel to the sides of the square) containing none of the points $z_1, \dots, z_n$ in the interior. Put

$$T(n) = \sup_{z \in Q^n} T(z). \tag{1}$$

In [Fr] Erich Friedman calculated $T(n)$ for $n \leq 2$, obtained the upper bounds for $n \leq 9$ and posed the question about the value of $T(n)$ for arbitrary n. The aim of this note is to give a partial answer to the Friedman's question.

We remark that the function $T(z)$ is continuous on Q^n for each n and hence the supremum in (1) holds for some $z \in Q^n$.

1. Lower bounds

Below we shall often use the following simple lemma.

Lemma 1.1. *For every $n \geq 1$ there is a point $z = (z_1, \dots, z_n) \in Q^n$ such that $T(z) = T(n)$ and $z_i \in [T(n); 1 - T(n)]^2$ for each i.*

Proof. Let $z = (z_1, \dots, z_n) \in Q^n$ and $T(z) = T(n)$. If there exist a point $z_i < T(n)$ then put $z' = (z_1, \dots, z_{i-1}, T(n), z_{i+1}, \dots, z_n)$. It is easy to check that $T(z') = T(n)$. The other cases are considered similarly. □

The sequences $\{z_1, \dots, z_n\}$ satisfying the conditions of Lemma 1.1 we shall call *regular*. Remark that a regular sequence has a point on every side of the square $[T(n); 1 - T(n)]^2$. This square we shall call the *basic square*.

Lemma 1.2. $T(n + m + 1) \geq T(n)T(m)/(T(n) + T(m))$ *for every $n, m \geq 0$.*

Proof. Let $z = (z_1, \dots, z_{n+m+1}) \in Q^{n+m+1}$ and $T(z) = T(n + m + 1)$. Divide Q by a vertical line l going through one of the points z_i such that there are n points to the left of l and m points to the right. Let l intersects the x-axis in the point x. Then $xT(n) \leq T(n + m + 1)$ and $(1 - x)T(m) \leq T(n + m + 1)$. Hence $T(n + m + 1)/T(n) + T(n + m + 1)/T(m) \geq x + 1 - x = 1$. □

Lemma 1.3. $T(m(n + 1) - 1) \geq T(n)/m$ *for every $n, m \geq 1$.*

Proof. Let $z = (z_1, \dots, z_{m(n+1)-1}) \in Q^{m(n+1)-1}$ and $T(z) = T(m(n + 1) - 1)$. As in Lemma 1.2 divide Q into m parts by vertical lines $l_0, l_1, \dots, l_m$ (with l_0 going through the point $(0, 0)$, l_m going through the point $(1, 0)$ and other lines going through some of the points z_i such that there are exactly n points between subsequent lines. Then one one these m rectangles has width not less than $1/m$ and hence $T(m(n + 1) - 1) \geq T(n)/m$. □

Corollary 1.4. $T(n) \geq T(m)(m + 1)/(n + m + 1)$ *for every $n \geq m \geq 1$. Therefore*

$$\lim_{n \to \infty} T(n)n = \sup_{n \in \mathbb{N}} T(n)(n + 1).$$

Proof. Since $n \leq \lfloor (n+m+1)/(m+1) \rfloor (m+1) - 1$ then Lemma 1.3 implies the claim. □

Now we are going to prove the lower bounds for small n.

Proposition 1.5. *Let t_3 be the smallest root of the equation $t^3 - 6t^2 + 5t - 1 = 0$. Then $T(3) = t_3$.*

Proof. Since in [Fr] is proved that $T(3) \leq t_3$ it suffice to show only the lower bound. Consider a regular sequence $z = z_1, \dots, z_3$ with $T(z) = T(3) = t$. Then there is at least one z_i in the corner of the basic square. One of the following cases is satisfied
1. There are no z_i in the interior of the basic square. Without loss of generality we may suppose that the points z_i have the coordinates (t,t), $(x, 1-t)$ and $(1-t, y)$. The following inequalities hold.

$$\begin{cases} x(1-t) \leq t \\ (1-t)y \leq t \\ (1-x)(1-y) \leq t \end{cases}$$

Thus $x, y \leq \frac{t}{1-t}$ and thus $(1 - \frac{t}{1-t})^2 \leq t$ which gives $t^3 - 6t^2 + 5t - 1 \geq 0$. The smallest root of the corresponding equation is 0.3079....
2. There is exactly one z_i in the interior of the basic square. Draw the vertical line through this point. It divides the bottom side of the square Q into two parts having the lengths not greater than $t/(1-t)$. Thus $2t/(1-t) \geq 1$ and $t \geq 1/3 > t_3$. □

Proposition 1.6. $T(4) = 1/4$.

Proof. Since in [Fr] is proved that $T(4) \leq 1/4$ it suffice to show only the lower bound. Consider a regular sequence $z = z_1, \dots, z_4$ with $T(z) = T(4) = t$. Then one of the following cases is satisfied
1. There are no z_i in the interior of the basic square. Then $(1-2t)^2 \leq t$ and thus $t \geq 1/4$.
2. There is exactly one z_i in the interior of the basic square. Then there is at least one z_i in the corner of the basic square. Without loss of generality we may suppose that the points z_i have the coordinates (t,t), $(x, 1-t)$, $(1-t, y)$ and (u, w). Then the next four cases are possible.
2a. $u \leq x$ and $w \leq y$. Then the following inequalities hold.

$$\begin{cases} (1-t)w \leq t \\ x(1-t) \leq t \\ (1-u)y \leq t \\ u(1-t) \leq t \\ (1-x)(1-y) \leq t \end{cases}$$

The first pair of inequalities gives $\frac{t}{1-t} + \frac{t}{x} \geq 1$. The second pair of inequalities gives $\frac{t}{1-t} + \frac{t}{y} \geq 1$. Hence $x, y \leq \frac{t-t^2}{1-2t}$ and $(1 - \frac{t-t^2}{1-2t})^2 \leq t$ which gives $1 - 7t + 15t^2 - 10t^3 + t^4 \leq 0$. The smallest root of the corresponding equation is $0.2831... > 1/4$.
2b. $u \leq x$ and $w \geq y$. Then the following inequalities hold.

$$\begin{cases} (1-2t)w \leq t \\ x(1-w) \leq t \\ (1-u)y \leq t \\ u(1-t) \leq t \\ (1-x)(1-y) \leq t \end{cases}$$

The first pair of inequalities gives $\frac{t}{1-2t}+\frac{t}{x}\geq 1$. The second pair of inequalities gives $\frac{t}{1-t}+\frac{t}{y}\geq 1$. Hence $x\leq\frac{t-2t^2}{1-3t}$ and $y\leq\frac{t-t^2}{1-2t}$. Thus $(1-\frac{t-t^2}{1-2t})(1-\frac{t-2t^2}{1-3t})\leq t$ which gives $1-8t+20t^2-16t^3+2t^4\leq 0$ since $t<1/3$. The smallest root of the corresponding equation is $0.2599... > 1/4$.
2c. $u\geq x$ and $w\leq y$. It is similar to Case 2a.
2d. $u\geq x$ and $w\geq y$. Then the following inequalities hold.

$$\begin{cases}(1-2t)w\leq t\\ (1-x)(1-w)\leq t\\ x(1-t)\leq t\end{cases}$$

The first two of inequalities give $\frac{t}{1-2t}+\frac{t}{1-x}\geq 1$. Since $x\leq\frac{t}{1-t}$ then $\frac{t}{1-2t}+\frac{t}{1-t/(1-t)}\geq 1$. Thus $t^2-4t+1\leq 0$ since $t<1/2$. The smallest root of the corresponding equation is $0.2679... > 1/4$.
Hence $T(z)\geq 1/4$ in Case 2.
3. There are exactly two z_i in the interior of the basic square. Draw the vertical lines through these points. They divide the bottom side of the square Q into three parts having the lengths not greater than respectively $t/(1-t)$, t, $t/(1-t)$. Thus $t^2-4t+1\leq 0$. The smallest root of the corresponding equation is $0.2679... > 1/4$. □

Corollary 1.7. $T(n)\geq 1/n$ *for every* $n\geq 4$.

Proof. The proof of the can be done by induction with Lemma 1.2, since

$$T(n+1)\geq\frac{T(n)}{1+T(n)}\geq\frac{n^{-1}}{1+n^{-1}}=\frac{1}{n+1}.$$

□

Remark 1.8. The following ideas we shall use in the computer calculation. Consider a regular sequence $z=z_1,\dots,z_n$ with $T(z)=T(n)=t$. Draw the vertical line through each of z_i. These lines divide the bottom side of the square Q into $n+1$ parts with length of each part between 0 and t. The union of two parts containing the projection of a point z_i we shall call the *segment corresponding* to z_i. If the point z_i lies in the horizontal side of the basic square then the length of the segment corresponding to z_i is not greater than $t/(1-t)$. Let S_1 and S_2 be the corresponding segments to the different points from the horizontal side of the basic square. If the segments S_1 and S_2 don't intersect then $t(n-3)+2t/(1-t)\geq 1$. Suppose that the segments S_1 and S_2 intersect. Then the length of it's intersection is not greater than $t/(1-2t)$. The rest $n-2$ segments are divided into two parts: k segments from the left to the intersection and $n-2-k$ segments to the right to it. Let

$$M=\max\left\{\frac{1}{T(n-3)},\frac{1}{T(0)}+\frac{1}{T(n-4)},\frac{1}{T(1)}+\frac{1}{T(n-5)},\dots,\frac{1}{T(n-4)}+\frac{1}{T(0)},\frac{1}{T(n-3)}\right\}.$$

Then $t/(1-2t)+Mt\geq 1$.

Corollary 1.9. *The Remark 1.8 and the computer calculation yields the following bounds.*
$T(5)\geq(3-\sqrt{3})/6=0.21132....$
$T(6)\geq(7-\sqrt{17})/16=0.17980....$
$T(7)\geq(7-\sqrt{33})/8=0.15692....$

Remark 1.10. Author has the complicated proofs that $T(7)\geq 0.1575138805$, $T(8)\geq 0.1424523558$, $T(9)\geq 0.1293157040$ and that $\lim_{n\to\infty}nT(n)\geq 1/(\ln 2)=1.442695040...$ but I hope to improve these bounds by improving value of $T(n)$ for small n. I'm expecting that somebody can beat these bounds.

2. Upper bounds

In this section we modify the solution of a problem from the All Soviet Union Math Competition in year 1965 (see [EV] or [Sch]).

Theorem 2.1. *$T(k2^{k-1}) \le 2^{-k}$ for each $k \ge 1$.*

Proof. Let $0 < x < 1$ be a binary rational number. Then x has the unique representation $x = m2^{-r}$ where m is odd and r is a positive integer. This r we shall denote as $\deg x$. Put $A_k = \{(x, y) \in (0; 1)^2 : \deg x + \deg y = k + 1\}$. We claim that the set A_k intersects every unrotated rectangle $R \subset Q$ of area not less than 2^{-k}. The claim can be proved by the induction. For $k = 1$ the set A_k consists of one point $(1/2; 1/2)$ and the claim is true. Suppose that we showed the claim for all numbers less than k. As π_x and π_y we denote the respective projections from $Q = I^2$ onto the coordinate axis. Let $R \subset Q$ be an unrotated rectangle not intersecting the set A_k. Suppose first that $\pi_x(R) \ni 1/2$. Then $\pi_y(R)$ has length less than 2^{1-k}, since the distance between two subsequent points $(1/2, x_1)$ and $(1/2, x_2)$ from A_k is 2^{1-k}. Hence if $\pi_x(R)$ has length less than $1/2$ then we are done. In the opposite case the set $\pi_x(R)$ contains one of the segments $[1/4; 1/2]$ or $[1/2; 3/4]$. Let it be the segment $[1/4; 1/2]$. But the distance between two subsequent points in $\pi_y(A_k \cap [1/4; 1/2] \times I)$ is 2^{-k} and hence $\pi_y(R)$ has length less than 2^{-k}. If $\pi_x(R) \not\ni 1/2$ then the set R is contained in one of the segments $[0; 1/2]$ or $[1/2; 1]$. Let it be the segment $[0; 1/2]$. But $A_k \cap (0; 1/2) \times I = (1/2)A_{k-1}$ and hence by the inductive conjecture the area of R not exceeds $(1/2) * 2^{1-k} = 2^{-k}$. Since the set A_k has size $k2^{k-1}$ this completes the proof. □

We can use the construction from the above theorem to obtain the following recurrent upper bound of $T(n)$.

Proposition 2.2. *$T(2n + \lfloor 2/T(n) \rfloor) \le T(n)/2$.*

Proof. Let $T_n \subset Q$ be the set of n points intersecting every unrotated rectangle $R \subset Q$ of area greater than $T(n)$. Put $m = \lfloor 2/T(n) \rfloor$. There is a set $A = \{x_1, \dots, x_m\} \subset I$ with $x_1 \le T(n)/2$, $x_m \ge 1 - T(n)/2$ and $|x_i - x_{i+1}| \le T(n)/2$ for others i. Then the set $T = (1/2)T_n \cup (1/2)(T_n + 1) + 1/2 \times A$ intersects every unrotated rectangle $R \subset Q$ of area greater than $T(n)/2$ and $|A| = 2n + m$. □

Corollary 2.3. *$T(4n) \le T(n)/2$ for $n \ge 4$.* □

Proof. By Corollary 1.7 $T(n) \ge 1/n$ for $n \ge 4$. □

References

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