ON GRAPHS WITHOUT 4-CYCLES

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In [Fr] Erich Friedman posed the following

Question. What is the maximum number $E(n)$ of edges an n-vertex graph can have so that the resulting graph contains no 4-cycle?

Let G be an n-vertex graph and M_G be the corresponding matrix of the graph G. Then the graph G contains no 4-cycles iff the matrix M_G contains no rectangles with unit vertices.

Proposition 1. $E(n) \le \frac{n+n\sqrt{4n-3}}{4}$.

Proof. Let G be an n-vertex graph without 4-cycles. Let l_i be the number of units in the i-th row of the matrix M_G. There are $\binom{n}{2}$ possible pairs of columns of the matrix M_G. We say that a row i *forbids* a pair of columns $\{j,k\}$ iff $m_{ij}=m_{ik}=1$. Then the i-th row forbids $\binom{l_i}{2}$ pairs and therefore $\sum\binom{l_i}{2} \le \binom{n}{2}$. Let $2E=\sum l_i$. Using the inequality between an arithmetic mean and a quadratic mean we obtain $4E^2-2nE \le n^2(n-1)$ which implies $E \le \frac{n(\sqrt{4n-3}+1)}{4}$.

Proposition 2. *If $n=q^2+q+1$ where q is a power of a prime then* $E(n) \ge \frac{(n-1)(\sqrt{4n-3}+1)}{4} - 1$.

Proof. Let p be a prime, $q=p^k$ and F_q be the finite field of order q. Consider the finite projective plane $\pi = P(2,q)$. Let π_0 be the three dimensional vector space over F_q. Define an equivalence relation on $\pi_0\backslash\{0\}$ putting $x\sim y$ iff there exists $\lambda\in F_q\backslash\{0\}$ such that $x=\lambda y$. Put now $\pi=\pi_0\backslash\{0\}/\sim$. It is well known and easy to see that $|\pi| = q^2+q+1$. Let $\{x_i\}$ be an enumeration of π. Define a matrix M putting

$$m_{ij} = \begin{cases} 1, & (x_i,x_j)=0 \wedge i\ne j \\ 0, & otherwise. \end{cases}$$

Then M is a symmetric matrix and therefore $M=M_G$ for some $|\pi|$-vertex graph G.

It is easy to check that the matrix M_G contains no rectangles with unit vertices. Consider sets $A_1=\{(1,x_2,x_3) : x_2,x_3\in F_q\}$, $A_2=\{(0,1,x_3) : x_3\in F_q\}$, $A_3=\{(0,0,1)\}$. Then $A=\bigcup A_i$ is a full family of representatives of the relation $\sim$. In the following table the value at i-th row and j-th column is $|\{x\in A_i, y\in A_j : (x,y)=0\}|$.

q^3-q	q^2	q
q^2	$q-1$	1
q	1	0

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Now we estimate a number N of vectors $x \in A$ such that $(x, x) = 0$. If $p = 2$ then a map $x \to x^2$ is a bijection of F_q. Therefore an equation $x^2 = a$ has an unique solution for every $a \in F_q$. Hence in this case $N = q + 1$. If $p \neq 2$ then for all $x \in F_q$ holds $x^2 = (-x)^2$ and easy to show that in this case $N \leq q + 3$.[1].

By construction of M the graph G contains an least

$$\frac{q^3 - q + 2q^2 + 2q + q - 1 + 2 - N}{2} \geq \frac{q^3 + 2q^2 + q}{2} - 1 = \frac{(n-1)(\sqrt{4n-3}+1)}{4} - 1$$

edges.

As $\pi(n)$ we denote the number of prime numbers which are not greater than n.

Lemma 1. *[Ro] For $n \geq 55$ holds $\frac{n}{\ln n + 2} < \pi(n) < \frac{n}{\ln n - 4}$.*

Lemma 2. *For every natural $n \geq 2$ there exists a prime number $p \in [n - 6n/\ln n; n]$.*

Proof. For $n \leq 2 \cdot 10^5$ the lemma is verified by computer calculation. If $n > 2 \cdot 10^5$ then $n - 6n/\ln n \geq 55$ and $\ln n > 12$. Therefore $\ln(n - 6n/\ln n) > \ln n - 6/\ln n - (6/\ln n)^2$. Hence Lemma 1 implies that

$$\pi(n) - \pi(n - 6n/\ln n) > \frac{n}{\ln n + 2} - \frac{n - 2n/\ln n}{\ln(n - 6n/\ln n) - 4} > \frac{n}{\ln n + 2} -$$

$$\frac{n - 6n/\ln n}{\ln n - 6/\ln n - (6/\ln n)^2 - 4} = \frac{6n(1 - 6n/\ln n)}{\ln n(\ln n + 2)(\ln n - 6/\ln n - (6/\ln n)^2 - 4)} > 0.$$

Corollary. $E(n) = \frac{n^{3/2}}{2}\left(1 + O\left(\frac{1}{\ln n}\right)\right)$.

References

[A] Айгнер М., Комбинаторная теория. - М.: Мир, 1982.

[EV] Васильев Н.Б., Егоров А.А., Задачи всесоюзных математических олимпиад. - М.: Наука, 1988.

[Fr] *http://www.stetson.edu/~efriedma/mathmagic/1000.html.*

[Ro] Rosser B., Proc. London Math. Soc, 1939, v.45(2), p. 21-44.

[1] Using the Galua fields theory it is possible to obtain somewhat better estimation of N