SATURATED SETS OF DISTINCT INTEGER EQUILATERAL TRIANGLES

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Abstract. We prove that a saturated set of n equilateral triangles with distinct integer side lengths exists only for even n ≥ 10.

Introduction

An arrangement of triangles in the plane is called *saturated* if the intersection of any two is either empty or is a common vertex and every vertex is shared by exactly two triangles. While others have dealt primarily with saturated sets of congruent triangles [1-4], our focus is on equilateral triangles with integer side lengths that are all distinct. The saturated sets are straightforward to build for n ≥ 12, but much more difficult for n = 10. Before proceeding to the constructions, we prove a key result.

Theorem 1. In any saturated set of n equilateral triangles, n ≥ 10.

Proof. Every triangle has three vertices, each of which is required to be on exactly two triangles, giving a total number of vertices equal to 3n/2. Thus, n must be even. Consider the exterior boundary of a saturated configuration. Call triangles that share a side with the boundary *exterior triangles* and call the others *interior triangles*. Since every exterior triangle must have its third vertex (which we call an *inside vertex*) inside the boundary, the interior angles of the boundary must all be larger than 120º, two 60º angles plus whatever positive angle is between those triangles. Therefore, there are at least seven exterior triangles.

We now show that there are at least two interior triangles. Two triangles cannot share more than one vertex, so an exterior triangle cannot have its inside vertex shared by an adjacent exterior triangle. If non-adjacent exterior triangles never share inside vertices, then there must be at least $7/3 \geq 2$ interior triangles to share them. If non-adjacent exterior triangles share inside vertices, this splits the inside into at least two regions, each of which requires at least one interior triangle.

Because any saturated set needs at least seven exterior triangles and at least two interior triangles, and because n must be even, we have $n \geq 10$. ■

A saturated set of 10 distinct equilateral triangles

The configuration of four equilateral triangles shown in Figure 1 is a fundamental building block for our construction, which includes two copies of the configuration, turned on their sides and joined by a pair of equilateral triangles in the center. The variables labeling the various triangles denote their side lengths. We refer to the configuration in Figure 1 as a *4-tuple* of side lengths{a,b,c,d}. We define the *width* w of {a,b,c,d} as the distance |UV|.

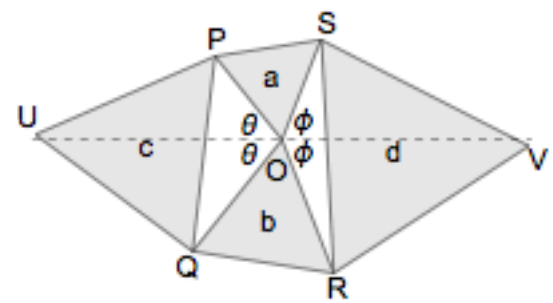

Figure 1. A configuration of 4 equilateral triangles.

Theorem 2. The width w is given by $w^2 = 4\,[(c^2+d^2) - (a^2+b^2)]$.

Proof. Refer to Figure 1. The origin O in the x-y plane is chosen as the point of intersection of the triangles POS (side a) and QOR (side b). The x-axis, shown as a dotted, directed line, bisects $\angle POQ$ and $\angle ROS$: thus, $\theta + \phi = 120^\circ$. We first prove that O is the midpoint of the line UV. From Figure 1, we obtain the coordinates for $P = (-a \cos \theta, a \sin \theta)$, $Q = (-b \cos \theta, -b \sin \theta)$, $R = (b \cos \phi, -b \sin \phi)$, and $S= (a \cos \phi, a \sin \phi)$.

Given arbitrary points (x_1,y_1) and (x_2,y_2), the two points that form equilateral triangles with them have coordinates $(x_\pm,y_\pm) = (\tfrac{1}{2}(x_1+x_2) \pm \tfrac{1}{2}\sqrt{3}\,(y_2-y_1),$ $\tfrac{1}{2}(y_1+y_2) \pm \tfrac{1}{2}\sqrt{3}(x_1-x_2))$. Using (x_-,y_-), we have

$$U = (-\tfrac{1}{2}(a+b)(\cos\theta + \sqrt{3}\sin\theta),\ \tfrac{1}{2}(a-b)(\sin\theta - \sqrt{3}\cos\theta)),$$

and using (x_+,y_+), we have

$$V = (\tfrac{1}{2}(a+b)(\cos\phi + \sqrt{3}\sin\phi),\ \tfrac{1}{2}(a-b)(\sin\phi - \sqrt{3}\cos\phi)).$$

Because $\phi = 120^\circ - \theta$, we see that $\cos\phi + \sqrt{3}\sin\phi = \cos\theta + \sqrt{3}\sin\theta$ and also that $\sin\phi - \sqrt{3}\cos\phi = \sqrt{3}\cos\theta - \sin\theta$. We conclude that $U = -V$ and therefore that O is the midpoint of the line from U to V. Consequently, the distance between U and O is ½ w, leading to:

$$w^2 = [(a+b)(\cos\theta + \sqrt{3}\sin\theta)]^2 + [(a-b)(\sin\theta - \sqrt{3}\cos\theta)]^2.$$

Expanding the right-hand side, using the identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, and then simplifying, we obtain

$$w^2 = 4(a^2 + b^2) + 4ab(\sqrt{3}\sin 2\theta - \cos 2\theta).$$

From triangle POQ, we have $c^2 = a^2 + b^2 - 2ab\cos 2\theta$ and from triangle ROS, we have $d^2 = a^2 + b^2 - 2ab\cos 2\phi$ by applying the Law of Cosines. With $\phi = 120^\circ - \theta$, we obtain $\cos 2\phi = -\tfrac{1}{2}\cos 2\theta - \tfrac{1}{2}\sqrt{3}\sin 2\theta$, yielding

$$(c^2 + d^2) - (a^2 + b^2) = a^2 + b^2 + ab(\sqrt{3}\sin 2\theta - \cos 2\theta),$$

and thus $w^2 = 4[(c^2 + d^2) - (a^2 + b^2)]$. ■

From Theorem 2, we see that if a, b, c, and d are integers, then w^2 is also an integer. However, the width w is not necessarily an integer.

Theorem 3. The formula $3(a^4 + a^2b^2 + b^4) + (c^4 + c^2d^2 + d^4) = 3(a^2+b^2)(c^2+d^2)$ holds for any 4-tuple {a,b,c,d} characterizing a configuration of 4 equilateral triangles as depicted in Figure 1.

Proof. Referring to Figure 1, we let $2\theta = 120^\circ - \psi$ and $2\phi = 120^\circ + \psi$. From triangle POQ, we obtain $c^2 = a^2 + b^2 + ab(\cos\psi - \sqrt{3}\sin\psi)$ and from triangle ROS, $d^2 = a^2 + b^2 + ab(\cos\psi + \sqrt{3}\sin\psi)$, using the Law of Cosines. Thus, $c^2 + d^2 = 2(a^2 + b^2 + ab\cos\psi)$ and $d^2 - c^2 = 2ab\sqrt{3}\sin\psi$. We solve these equations for $\cos\psi$ and $\sin\psi$, respectively, then square and add them to eliminate ψ. Simplifying the resulting equation yields the desired result. ■

We discovered 4-tuples {a,b,c,d} satisfying the formula in Theorem 3 via a comprehensive computer search that calculated d for given a, b, and c. We say that a 4-tuple is *primitive* if a, b, c, and d are co-prime. Among all the 4-

tuples we found, the only primitive one having integer width is X = {323, 392, 407, 713}, with width 1290. Our saturated set of 10 distinct equilateral triangles starts with the 4-tuple Y = {12369, 12776, 21293, 22231}, chosen because $a \approx b$ and $c \approx d$, allowing it to fit between two end configurations without causing any triangles to overlap. We scale up Y by a factor of 1290, and then replace the two larger triangles with 21293X and 22231X to serve as the end configurations. Figure 2 displays the result.

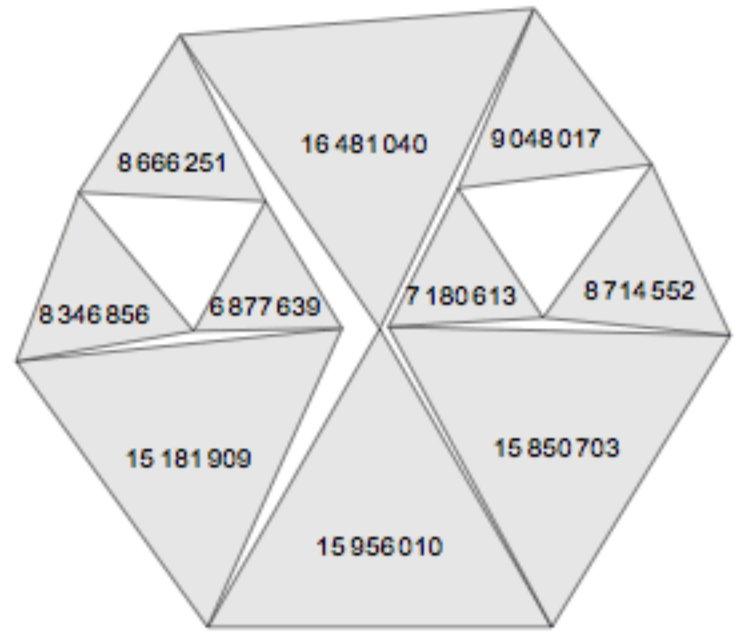

Figure 2. A saturated set of 10 distinct equilateral triangles.

Saturated sets for even n > 10

It is much easier to construct examples of saturated sets of n distinct integer equilateral triangles for even $n > 10$. Let the 4-tuple Z = {35, 139, 146, 169}. If $n = 0 \pmod 4$, then string together a loop of n/4 copies of Z scaled up by factors of 2, 3, 4, ... If $n = 2 \pmod 4$, then start with triangles of sides 150 and 250 touching at a vertex and tilted appropriately such that two of the free vertices connect precisely with the 4-tuple Z. Connect the other two free vertices by stringing together (n–6)/4 copies of Z scaled up by factors of 2, 3, 4 ….

Open questions

One can ask for which even n there exist saturated sets of n equilateral triangles of k different integer sizes. We denote any such structure by (n,k). For $k = 1$, there are examples for $n \geq 42$ [4], and 42 is strongly suspected to be minimum. For even $n \geq 10$ and $2 \leq k \leq n$, we conjecture that (n,k) exists except for (10,2), (10,3), and (14,2).

References

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