Maximizing Angle Counts for n Points in a Plane

By

Brian Heisler

A SENIOR RESEARCH PAPER PRESENTED TO THE DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE OF STETSON UNIVERSITY IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF BACHELOR OF SCIENCE

STETSON UNIVERSITY

2008

ACKNOWLEDGMENTS

Thanks to God foremost for providing me with the ability to study mathematics and to have the opportunity to research an area of academics which I have grown to love throughout my life.

Thank you to Dr. Erich Friedman, who has provided me with the opportunity to work with him throughout the year. With his guidance, I have been able to indulge into the mathematics of Combinatorial Geometry, enhancing my understanding of angles, and the properties which allow them to be formed.

Thanks to fellow math majors Kevin Heisler and William Wood for allowing me run off ideas from the top of my head and to provide insight as to whether or not my theories were mathematically correct.

Thanks to Dr. Hari Pulapaka for teaching me aspects of Graph Theory, which I have used in my research of graphing n points in the plane. I have found various theories in this branch to be useful in bounding the maximum angle count for n points in a plane.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURE

ABSTRACT

MAXIMIZING ANGLE COUNTS FOR N POINTS ON A PLANE

By

Brian Heisler

May 2009

Advisor: Dr. Erich Friedman
Department: Mathematics and Computer Science

Given a number of points, n, to be plotted on an infinite, two-dimensional plane, we will attempt to maximize the number of times angle Θ can be constructed using those n points. For the purpose of this proposal, we will only be considering the angles $\Theta \leq 180^\circ$. For collinear points, we allow $\Theta = 0^\circ$ and $\Theta = 180^\circ$, as these angles do exist on a straight line.

Define $f(n, \Theta)$ to be the function whose value represents the maximum angle count of Θ on n points. As n becomes increasingly large, it is less likely that we will be able to find the exact values of $f(n, \Theta)$, and will thus attempt to find the upper and lower bounds for $f(n, \Theta)$. The lower bounds are the greatest number of angles that have been counted from a concrete construction of n points in a plane and the upper bounds are the theoretical greatest number of angles that can be constructed from n points in a plane.

Chapter 1. Introduction

Given n points in a plane, there is a graph, G, which can be constructed and which consists of x number of angles Θ. The purpose of this research is to find the maximum number of occurrences of Θ, or maximum angle count. For the trivial case $n = 3$, finding the maximum angle count is simple. For $n \geq 4$, finding the maximum angle count is not as simple. For these cases, we must bound the maximum angle count with upper and lower limits based upon mathematical theory and proofs.

Define an *angle count* to be the number of times an angle Θ can be constructed for a given number of points.

Define $f(n, \Theta)$ to be maximum angle count for those n points on the given angle Θ. Given the complexity of the construction of angles, we will not be able to give exact values for the maximum angle count for $n \geq 4$. Instead, we will determine an upper and lower bound between which the true value for the maximum angle count lies.

Chapter 2. Small N Values

2.1. N = 3

We will start with the simplest case, when N = 3:

Theorem 1:

$$
f(3, \Theta) = \begin{cases} 3 & \text{if } \Theta = 60^\circ \\ 2 & \text{if } 0^\circ \leq \Theta < 60^\circ \text{ or } 60^\circ < \Theta < 90^\circ \\ 1 & \text{if } 90^\circ \leq \Theta \leq 180^\circ \end{cases}
$$

Proof:

Case 1: $f(3, 60^\circ) = 3$

Consider Figure 1, an equilateral triangle ABC. By definition, an equilateral triangle has 3 congruent sides, and thus 3 congruent angles. The sum of the interior angles of a triangle is 180˚. Let Θ represent each of the 3 congruent angles in the equilateral triangle ΔABC. Then $3\Theta = 180^\circ$. So $\Theta = 60^\circ$.

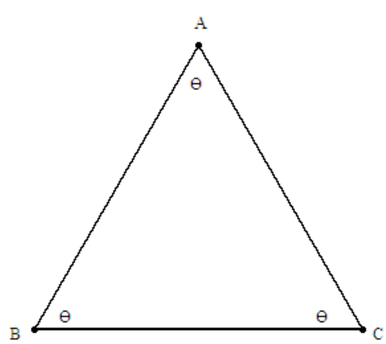

Figure 1

Case 2: $f(3, \Theta) = 2$ for $0^\circ \leq \Theta < 60^\circ$ or $60^\circ < \Theta < 90^\circ$

Consider Figure 2, the isosceles triangle ABC. Each of the base angles, $\angle ABC$ and $\angle ACB$ are congruent and acute (property of isosceles triangles). Let Θ represent $\angle ABC$. Since side AB and side AC are congruent, their opposing angles are congruent. So $\angle ACB = \Theta$. Let α represent $\angle BAC$. Then $\Theta + \Theta + \alpha = 180^\circ$. Then $\Theta < 90^\circ$ is the solution for Θ for this equation. Clearly then Θ cannot occur more than 2 times for the specified value.

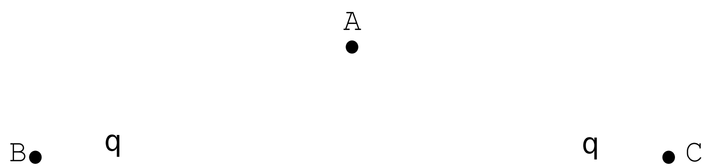

Figure 2

Case 3: $f(3, \Theta) = 1$ for $90^\circ \leq \Theta \leq 180^\circ$

Consider Figure 3, the triangle formed from 3 collinear points A, B, and C. This triangle can be viewed as a "flattened" version of Figure 2. So angles ABC and ACB are both congruent and equal 0°. So $\angle BAC = 180^\circ$. By the same mathematical basis, $f(3, 0^\circ) = 2$. Let $90^\circ < \Theta < 180^\circ$. Consider Figure 2 again, the isosceles triangle ABC. We know the sum of the interior angles of a triangle is 180°. If $\Theta > 90^\circ$ then $2\Theta > 180^\circ$. So for $90^\circ \leq \Theta < 180^\circ$, $f(3\ \Theta) = 1$.

Figure 3

Let $\Theta = 180^\circ$. Then by the proof used in case 2, $f(3, 180^\circ) = 1$.

Let $\Theta = 90^\circ$. Then Θ occurs at the intersection of two perpendicular lines (Figure 4a. 4b, 4c, 4d). To name an angle, we need 3 points, with the middle point being the vertex from which the angle is subtended. So place one point, the vertex, at the intersection of the two perpendicular lines, and label it A. We must now form two more right angles using only two more points. Figures 4a, 4b, 4c, and 4d show the four different results from adding two more points on the graph (excluding the two cases where a straight angle is formed). From the four diagrams, we can see that creating two right angles using three points is not mathematically possible. So f(3, 90˚) = 1.

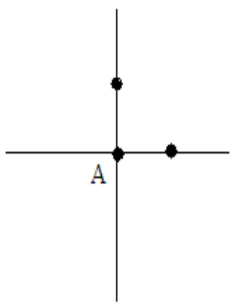

Figure 4a

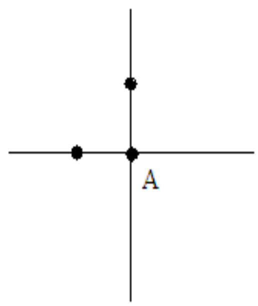

Figure 4b

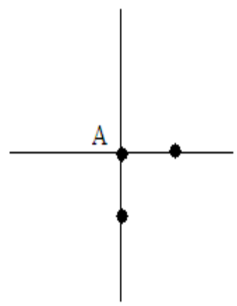

Figure 4c

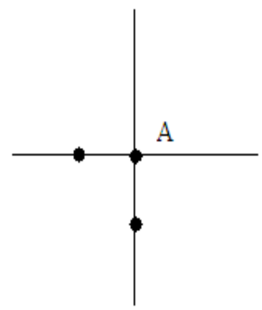

Figure 4d

2.2. N = 4

We next consider N = 4. First we will present a general result:

Theorem 2: The maximum number of triangles which can be constructed from n points in a plane is ${}_nC_3 =$ 3! – 3

Proof:

Given n points, we choose any 3 of the n to construct a triangle. Thus, we use the formula ${}_nC_3$ to solve for the maximum number of triangles that can be constructed using n points.

♣

Theorem 3: f(n, ϴ) ≤ f(3, ϴ) x (${}_nC_3$) for 0˚ ≤ ϴ ≤ 180˚ and n ≥ 4.

Proof:

In theorem 2, we showed the maximum number of triangles that can be formed from n points is ${}_nC_3$. In theorem 1 we showed the values of f(3, Θ). So we can have, at most, f(3, Θ) angles in each triangle and ${}_nC_3$ triangles. So f(n, Θ) = f(3, Θ) x (${}_nC_3$). ♣

We can now use theorems 1, 2, and 3 to find maximum angle counts for Θ on 4 points in a plane. For some values of Θ, though, we will bound f(4, Θ) with an upper and lower limit, as the exact maximum angle count is not apparent.

Theorem 4: For $0^\circ < \Theta < 90^\circ$, $4 \leq f(4, \Theta) \leq 8$.

Proof:

Consider Figure 5, the quadrilateral ABCD formed by vertically reflecting Figure 1 about its base side. We now have 4 equivalent angles $\Theta < 90^\circ$. From theorems 2 and 3, we know there are at most 4 triangles that can be constructed from 4 points, and can have a maximum of 4f(3, Θ) angles for $\Theta < 90^\circ$. So $f(4, \Theta) \leq 8$. ♣

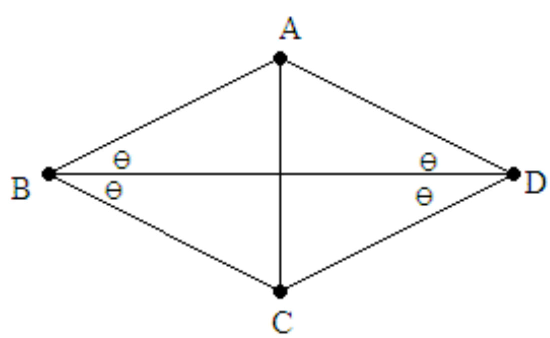

Figure 5

Theorem 5: For $90^\circ < \Theta < 180^\circ$, $2 \leq f(4, \Theta) \leq 4$.

Proof:

Consider Figure 6, a 90˚ clockwise rotation of Figure 5. If $\angle ADB$ and $\angle ABD < 45˚$, then $90˚ < \angle DAC < 180˚$ and $90˚ < \angle DCB < 180˚$. So $f(4, \Theta) \geq 2$ for $90˚ < \Theta < 180˚$. From theorems 2 and 3 we know that we can have a maximum of $4f(3, \Theta)$ obtuse angles. So $f(4, \Theta) \leq 4$ for $90˚ < \Theta < 180˚$. ♣

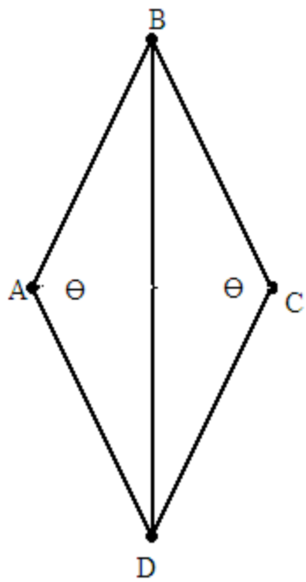

Figure 6

Theorem 6: $f(4, 90˚) = 4$.

Proof:

Let ABCD be a square, with the four angles $\angle ABC$, $\angle BCA$, $\angle CDA$, and $\angle DAC$ equivalent right angles (see Figure 7). We now have a concrete example to show $f(4, 90˚) \geq 4$. From theorem 1, $f(3, \Theta) = 1$ and from theorem 3, the maximum angle count for $90˚ \leq \Theta < 180˚$, $f(4, \Theta) \leq 4$. Thus, $4 \geq f(4, 90˚) \geq 4$. So, $f(4, 90˚) = 4$. ♣

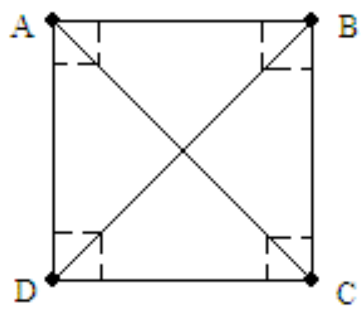

Figure 7

Theorem 7: f(4, 0˚) = 8.

Proof:

From theorem 1, f(3, 0˚) = 2. Consider Figure 8, the graph of 4 collinear points ABCD. From theorem 2, there are 4 unique triangles which can be constructed. So, there are 4f(3, 0˚) = 8 angles Θ = 0˚. ♣

Figure 8

Theorem 8: f(4, 180˚) = 4.

Proof:

In theorem 1, we proved that f(3, 180˚) = 1. Consider again Figure 8. By the same calculations proven in case 2 of theorem 1, there are 4f(3, 180˚) = 4 angles Θ = 0˚.♣

Theorem 9: f(4, 45˚) = 8

Proof:

In theorem 3 we proved that for Θ < 90˚, 4 ≤ f(4, Θ) ≤ 8. Consider Figure 7, the square ABCD. Each of the vertex angles are right angles. The diagonals of the square bisect each of the vertex angles, creating two 45˚ angles at each one. So there are eight angles of 45˚. So we have an figure of f(4, 45˚) ≥ 8, and we know that f(4, 45˚) ≤ 8. So f(4, 45˚) = 8. ♣

Theorem 10: 6 ≤ f(4, 60˚) ≤ 8

Proof:

Consider Figure 9, the quadrilateral formed from the reflection of and equilateral triangle. We now have two equilateral triangles, each having f(3, 60˚) = 3. So f(4, 60˚) ≥ 6. From theorem 3, we know f(4, 60˚) ≤ 8. ♣

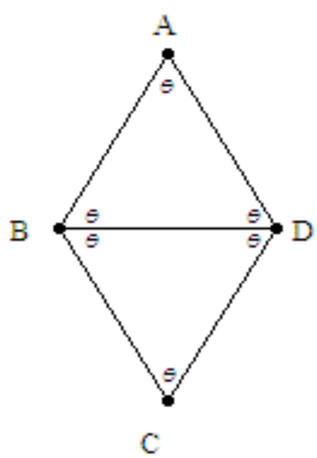

Figure 9

Theorem 11: 3 ≤ f(4, 120˚) ≤ 4

Proof:

Consider Figure 10 below, the equilateral triangle ABC, with each angle bisected. The bisecting segments intersect at point D. Each line segment AD, CD, and BD all bisect the angles ∠ABD, ∠ACD, and ∠BAC. Because ABC is an equilateral triangle, these three line segments will intersect at the exact center of the triangle, and the 3 line segments will be congruent. Since the 3 line segments are equal, we have 3 congruent, isosceles triangles. So each of the 3 angles ∠ABD, ∠ACD, and ∠BAC are congruent and sum to 360˚. Let Θ represent each of the 3 angles. Then $3\Theta = 360˚$. So $\Theta = 120˚$. So $f(4, 120˚) \geq 3$. From theorem 3, $f(4, 120) \leq 4$. ♣

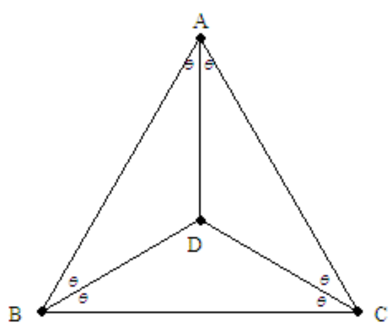

Figure 10

Chapter 3. Special Θ Values

Among the general proofs for the maximum angle counts are certain Θ values that can be considered exceptions to this general proof. The general proof is considered to be the method of bounding the maximum angle count $f(n, \Theta)$, where $n \geq 4$. Using this method, we simply maximize the number of triangles using theorem 2 and multiply this value by the max angle count found from $f(3, \Theta)$. This will give us an upper bound. The lower bound is found by construction of a figure on the n points. These special Θ values are considered special because they have different bounds, with the main difference being that their lower limit is greater than that of the lower limit of the general proof that Θ would fall into. Given a special Θ value for n points, $n \geq 4$, a particular bound results, which differs from the bound given by the general proof for those n points. This special Θ value does not have the same bound for a construction or graph on a different n value, $n \geq 4$.

3.1. The Unique Θ

In the field of mathematics called Graph Theory, a *complete graph* is defined as the simple graph on n vertices such that every edge is connected to all other edges, and is denoted K_n.

Theorem 12: Given n points, where $n \geq 3$ and no 3 are collinear, □ a unique angle, □ □ $180^{\circ} \bmod (n) = 0$, □ □ can occur no more than $n(n-2)$ times.

Proof:

Consider the graph of K_n in which all n points lie on the circumference of a circle, C (Figure 11). Since every vertex in the vertex set $V(K_n)$ is connected, K_n has the greatest number of edges for a graph on n points. This implies that K_n also has the greatest number of angles for a graph on n points. In K_n every vertex is connected to (n-1) vertices. So there are n(n-1)/2 edges. Since every vertex is connected to (n-1) other vertices, there are (n-1) edges connecting each vertex. This creates at most (n-2) copies of one unique angle □ at each vertex, each of which are interior angles. Thus there are n(n-2) copies of □ in the graph K_n. The sum of the interior angles of any polygon on n vertices equals (n-2)x180˚. So given n(n-2) equivalent copies of □, we have an equation □x(n)x(n-2) = (n-2)x180˚. Solving for □, we find □ = 180˚/n. ♣

Theorem 13: The central angle subtended by two points on a circle is twice the inscribed angle subtended by those points (The Central Angle Theorem) [B2].

Proof [B3]**:**

Case 1: One chord is a diameter [B3].

Let G be a circle with center C (see Figure 11a [A2]). Choose two points on the circle, say A and B. Draw line AC and extend this line until it intersects circle G at point D on the circumference. □DCB is a central angle and will be labeled ϴ. Connect points V and A with a chord, and label □CAB ψ. Lines CA and CB are both radii of the circle G, so triangle CAB is isosceles. By the properties of an isosceles triangle, the angles opposite the equivalent sides are congruent. So □ABC is congruent to □CAB, and will also be labeled ψ. □DCB and □DCA are supplementary angles, and thus add to 180˚. So □BCA = (180˚ - ϴ). We know the interior angles of a triangle add to 180˚ so we know ψ + ψ + (180˚ - ϴ) = 180˚. Solving for ϴ, we find ϴ = 2 ψ.

We will use this proof to help prove the Central Angle Theorem when the diameter is not a chord.

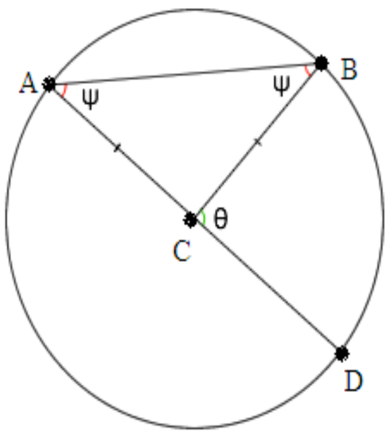

Figure 11a

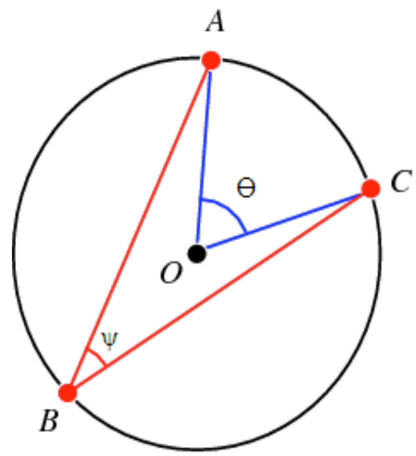

Figure 11b

Case 2: Center of the circle in the interior of the angle [B3].

Let C be a circle with center point O (see Figure 12). Choose three points, say A, B, and D on the circle C. Draw lines AB and AD. Now ∠DAB is an interior angle. Draw line VO and extend it until it intersects the circle at point E. Now ∠DAB subtends arc BD on the circle. ∠ DAE and ∠EAB are also now inscribed angles, each having one side passing through the center of the circle C. So, ∠DAB = ∠DAE + ∠EAB

Let ∠DAB = ψ_0

Let ∠DAE = ψ_1

Let ∠EAB = ψ_2

So, $\psi_0 = \psi_1 + \psi_2$ (1)

Draw lines DO and BO, so ∠DOB is a central angle, along with ∠DOE and ∠EOB.

So, ∠DOB = ∠DOE + ∠EOB

Let $\angle DOB = \Theta_0$

Let $\angle DOE = \Theta_1$

Let $\angle EOB = \Theta_2$

So, $\Theta_0 = \Theta_1 + \Theta_2$ (2)

We know from (A) that $\Theta_1 = 2\psi_1$ and $\Theta_2 = 2\psi_2$. Plugging these equations into (2), we find

$\Theta_0 = 2\psi_1 + 2\psi_2$

$\Theta_0 = 2(\psi_1 + \psi_2)$

$\Theta_0 = 2\ \psi_0$ (3) (Figure 12 [A2]).

Move point A along the circumference of G, and angle ψ_0 will never change. ♣

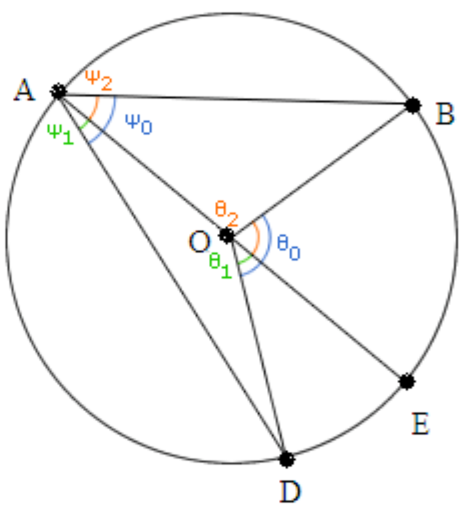

Figure 12

3.2. Special Values for N = 3

From theorem 1, for n = 3, $\Theta = 60^\circ$ is considered a special value since it can occur 3 times, in a graph of three points, whereas all other values $f(3, \Theta) \leq 2$. Since $\Theta = 60^\circ$ does not follow the general proof for n = 3 (i.e. $f(3, \Theta) = 2$ for $0^\circ \leq \Theta < 90^\circ$ and $f(3, \Theta) = 1$ for

$90^\circ \leq \Theta \leq 180^\circ$), it will not follow the general proof for $n > 3$ (i.e. $f(n, \Theta) \leq 2n$ for $0^\circ \leq \Theta < 90^\circ$ and $f(n, \Theta) \leq n$ for $90^\circ \leq \Theta \leq 180^\circ$) ♣.

Chapter 4. Future Work

Up until now, I have considered and examined the maximum angle counts for $n = 3$ and $n = 4$. As I continue my research, I will first continue examining the cases for $f(5, \Theta)$, and determine the bounds. While researching $n = 5$, and eventually $n > 5$, I will hope to come across a definite pattern between $f(n, \Theta)$ and $f(n+1, \Theta)$. Throughout my research, I was able to find more precise bounds for $f(n, \Theta)$, and will continue to search for even more precise ones than currently held.

To guide me in my research I will study some of the works of highly-renowned mathematician Paul Erdős. Erdős specialized in Graph Theory, focusing on the branch of random graphs and their properties. I am essentially dealing with random graphs in my research, as any graph can be constructed from n points, and I must consider a variety of these graphs. Also, Erdős published some of his research on the optimization of similar triangles, which I will consider. From theorem 2, it was first proposed to maximize the number of triangles in a graph on n points, and I believe his work will guide me in this idea.

References

A. Offline

[A1] J. Gross and J. Yellen, *Graph Theory and its Applications*, Edition 2, Chapman and Hall, Boca Raton, FL, 2006.

B. Online

[B1] http://mathworld.wolfram.com/images/eps-gif/CentralInscribedAngle_1000.gif - Google Image.

[B1] http://mathworld.wolfram.com/images/eps-gif/CentralInscribedAngle_1000.gif - John Page, *Central Angle Theorem*, Math Open Reference, 2007.

[B3] http://en.wikipedia.org/wiki/Inscribed_angle - *Inscribed Angle*, Wikipedia Web Site.

Biographical Sketch

Brian Heisler is a mathematics major who plans to fulfill the requirements for teacher's certification upon graduation. He has always enjoyed mathematics, from the time he was in elementary school and still today. After he graduates from Stetson, he will pursue his Master's degree in Education, and find a teaching position at a Florida high school. He hopes to teach either Algebra I or II, or Pre-Calculus, as numerical and algebraic mathematics are his favorite.

Sports have always been a major part of his life, and hopefully will remain as such in the future. Running has quickly grown to be both his best and favorite sport, starting the summer after his 7th grade year, and have continuing to this day. Currently, he runs for Stetson's cross country team as one of the top runners. He really enjoys competing in races, and has grown to find that distance is definitely his strong point in the sport.

One of his favorite hobbies is Texas Hold 'Em poker. He loves to play in large tournaments to test my skills and patience, and tries to play when he has free time. When he is not doing homework, running or playing poker, he enjoys spending time with his friends and family. He takes comfort in this time, knowing that he has people in his life who care about him and love him, and who will encourage him through anything he may face. It truly is a great feeling.