VERTEX PERFECT GRAPHS

By

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ABSTRACT

Vertex Perfect Graphs

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We explore a generalization of perfect numbers to connected graphs. Specifically, this paper defines vertex perfect graphs and investigates how these new objects are similar to and differ from their number theoretic counterparts.

The main results of this paper outline some of the basic properties of vertex perfect graphs in addition to producing tools useful for finding them. We prove that there are infinitely many vertex perfect graphs, and for most positive integer $n \leq 100$, we determine whether a vertex perfect graph of order n exists.

We additionally investigate the properties of vertex perfect trees. Focusing on the properties of their vertex divisors, we prove theorems that dictate the structure of these minimally connected graphs.

Finally, we further generalize this topic and define amicable graphs, new objects in graph theory that correspond to amicable pairs in number theory. We conclude by extending previous results on vertex perfect graphs to amicable graphs.

Chapter 1
Intro to Vertex Perfect Graphs

1.0 Introduction to Number Theory and Graph Theory

This section is a summary of fundamental concepts necessary for understanding the results stated later in this paper, namely perfect numbers and simple graphs. If the reader is already familiar with both of these topics, skipping this section is possible.

In number theory, a **divisor** q of a positive integer p is any number that satisfies $p/q \in \mathbb{Z}^+$. A **proper divisor** of p is any divisor of p that is less than p. If the sum of the proper divisors of p is less than p, we say that p is **deficient**. If the proper divisors of p sum to an integer greater than p, then p is said to be **abundant**. When p is the sum of its proper divisors, p is called **perfect**.

The number 6 is an example of a perfect number since 1, 2, and 3 sum to 6. 28 is also perfect as 1, 2, 4, 7, and 14 add to 28. An example of an abundant number is 12 since its proper divisors sum to 16, and any prime number is deficient as the only proper divisor of a prime is one.

We will now review some fundamental topics from the field of graph theory. In graph theory, a **graph** G consists of two sets: one of **vertices** and one of **edges**. Each edge connects a pair of vertices. If two vertices share an edge, they are said to be **adjacent**. The total number of vertices and edges of G are respectively referred to as G's **order** and **size**. The order of G is denoted by $|G|$ while the size of G is denoted by $|E_G|$. If in G no two vertices share multiple edges and no edges connect a vertex to itself, then G is said to be a **simple graph**.

All graphs can be visualized as a set of points (vertices) with lines connecting each point (edges). Take the following figure as an example: if we call the graph in the image G, then we can say that G is a simple graph with $|G| = 4$ and $|E_G| = 5$. The vertex v_1 is adjacent to all of the other vertices that compose G.

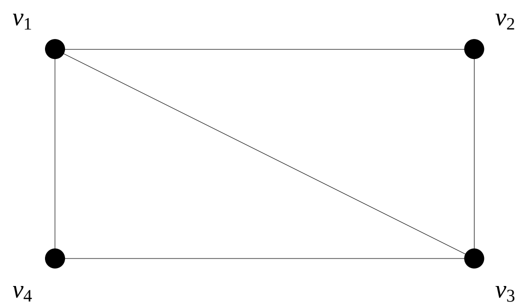

Figure 1. An example of a representation of a graph.

A **path** is an ordered listing of vertices of a graph G such that any vertex may only be included once and one vertex may only follow another if both are adjacent. An example of a path on the vertices of the graph in figure one could include v_1, v_3, and v_2.

A **subgraph** H of a graph G is a subset of vertices and edges from G. Any edge of G may be included in H as long as both of its corresponding vertices are included in H. A **component** of a graph G is any subgraph that includes any collection of vertices (and all of their corresponding edges) that can be joined together by a path. If a path cannot join two vertices in G, then those two vertices are said to be in separate components of G. If G has only one component, G is said to be **connected**. A **spanning subgraph** of a connected graph G is any subgraph that retains all of the vertices of G and is still connected.

As an example, the following figure 2 has 2 components, one formed by the labeled vertices 1 through 13, and one formed by the labeled vertices 14 through 18.

The subgraph of vertices labeled from 10 to 13 form a special type of graph called a **path graph**, a graph that is simply a path on n vertices (denoted by P_n). The subgraph formed by the set of vertices labeled 1 through 8 and all of their shared edges is also a special type of graph called a cycle. A **cycle** with order n is a path graph with the first and final vertex joined by an edge and (denoted C_n). The subgraph of vertices 1 through 9 and all of their shared edges is a **wheel graph** (denoted by W_n), a cycle in which all vertices are adjacent to some additional vertex. The vertices 14 through 18 and their shared edges form a **complete graph**, a graph in which every vertex is adjacent to every other vertex in the graph (denoted K_n). Finally, any connected subgraph of G that does not contain a cycle is a called **tree**.

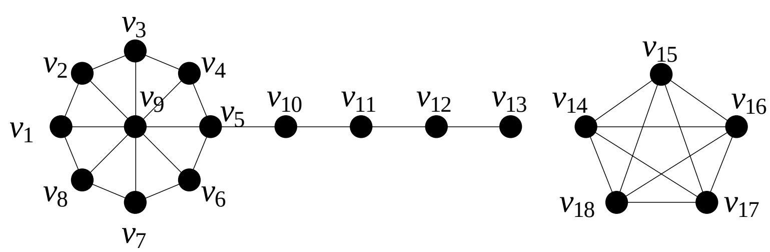

Figure 2. A graph with multiple components.

There are a number of operations that can be performed to alter a graph. The operation of **edge deletion** removes an edge between two vertices. The operation of **vertex deletion** deletes a vertex and any edge that it shares with another vertex. **Edge Contraction** erases a given edge and the pair of vertices that it connects, replacing them with a new vertex that is adjacent to all of the vertices that the previously pair were adjacent to. See the figure below for examples of these three operations.

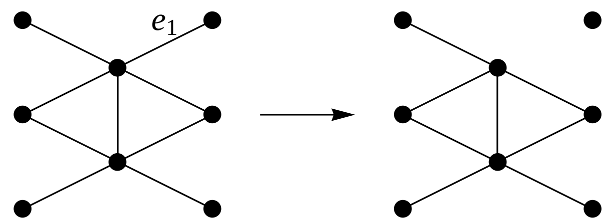

Figure 3a. An example of deleting an edge e_1 from a connected graph.

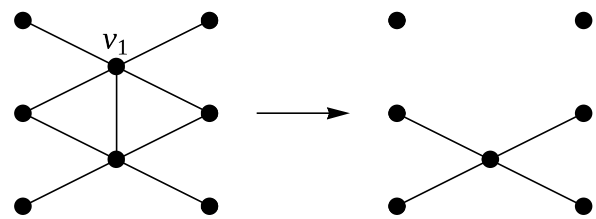

Figure 3b. An example of deleting a vertex v_1.

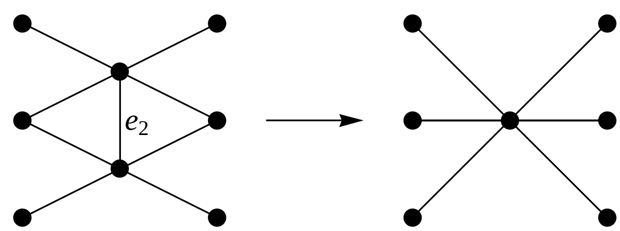

Figure 3c. An example of contracting an edge e_2.

Some final definitions concerning vertices are in order. The **degree** of a vertex v is the number of vertices that v is adjacent to in a simple graph. This is not the actual definition of the degree of a vertex in general, but because we will only be dealing with simple graphs in this paper, we stick to our less precise definition. The **maximum** and **minimum degrees** of a graph G are defined intuitively and are denoted as $\Delta(G)$ and $\delta(G)$, respectively. Revisiting figure 2, we can see that $\Delta(G) = 8$, which corresponds to the degree of vertex 9. $\delta(G) = 1$, and this degree corresponds to the vertex 13.

A **cut vertex** is any vertex that when deleted increases the number of components of a graph. The analog of a cut vertex in terms of edges is a **bridge**, which is any edge that when deleted increases the number of components of a graph. In figure 2, vertex 5 is a cut vertex and the edge that connects vertex 5 and vertex 10 is a bridge.

From this point on in the paper, it can be assumed that when we use the term graph, we are discussing simple graphs.

1.1 Vertex Perfect Graphs

The main focus of this research project concerns a generalization of perfect numbers to graphs. Specifically, the idea of a perfect number has been mapped to simple, connected graphs to form an analog with similar properties. We call these analogs vertex perfect graphs (VPG). To define VPGs, we will first need to generalize addition and division to simple, connected graphs.

We generalize addition by defining a new operation called **graph** covering. If a set of connected graphs $\{H_1, H_2, \ldots, H_n\} = S$ can have edges added between them (including possibly adding edges between two vertices in the same H_i) to produce a new connected graph G, then we say that G is **covered** by S. Similarly, if a connected graph G can have edges removed to produce S, G is covered by S. The following figure serves as an example of graph covering. The set S of three connected graphs seen in the left of the image covers the graph G seen at the right.

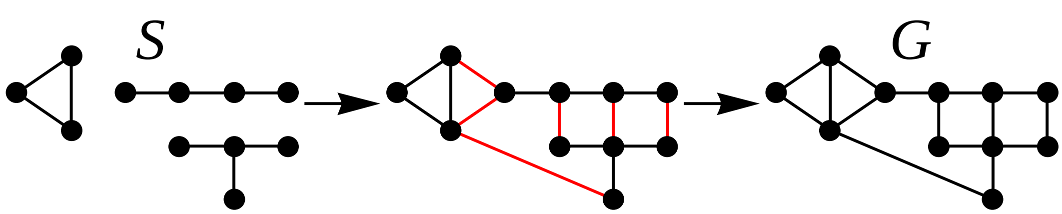

Figure 4. An illustration of a set S of graphs that cover G.

We now generalize division by defining a vertex divisor. If H and G are connected graphs, we say that H is a **vertex divisor** of G (or, for brevity, H is a **divisor** of G) if edges of G can be removed to yield $n \geq 2$ isomorphic copies of H. In other words, if G can be divided into n separate components of strictly H by edge deletion, then H is a divisor of G. The next figure illustrates how the graph H is a divisor of the three graphs that follow it.

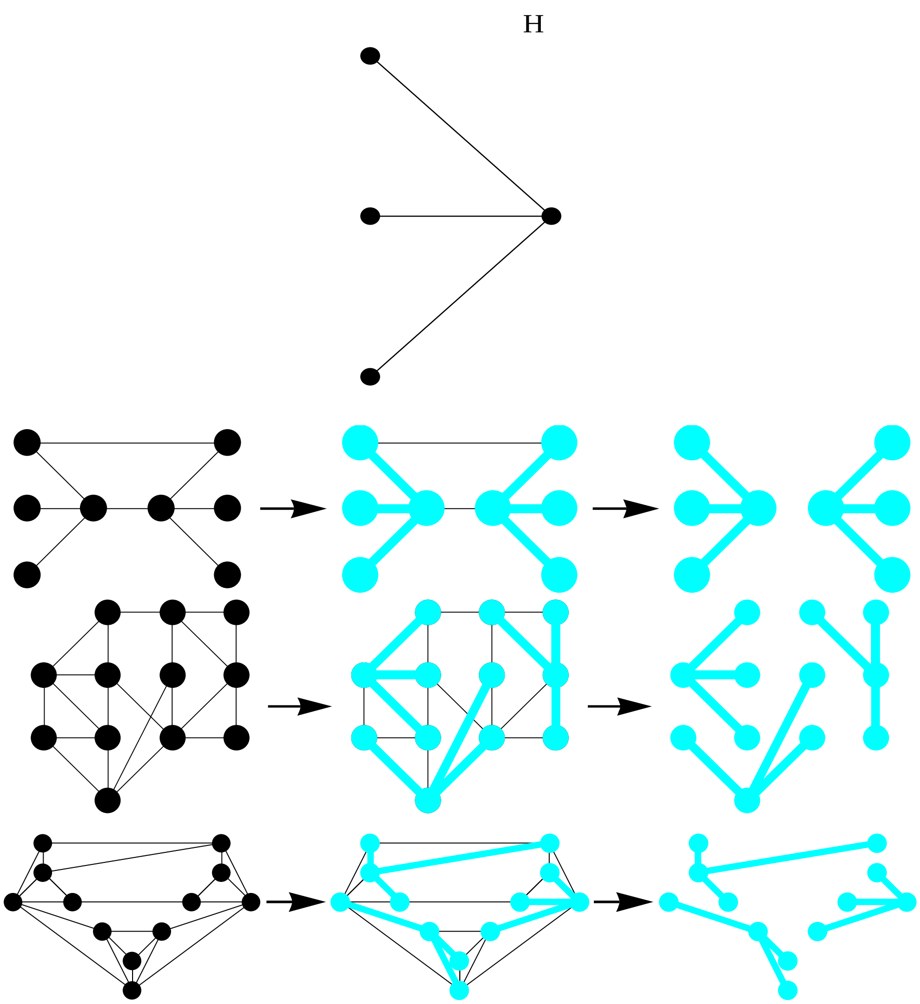

Figure 5. H as the divisor of various other graphs.

We are now ready to present the definition of a VPG. If a graph G is covered by the exhaustive set of its divisors, then G is **vertex perfect**. As an example, the following two path graphs on “familiar” orders (6 and 28, both perfect numbers) are shown with their perfect coverings.

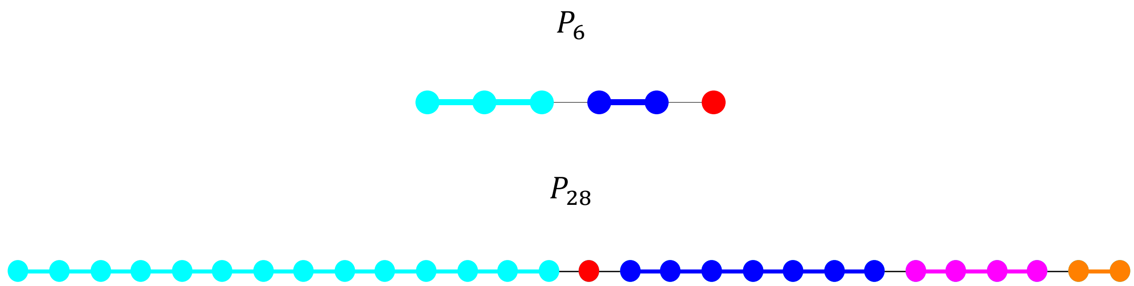

Figure 6. P_6 and P_{28}.

As the above figure suggests, there is a correspondence between paths with perfect orders being vertex perfect as well as the divisors of such graphs mapping to the proper divisors of perfect numbers. Other vertex perfect graphs also share similar relations that directly mirror their number theoretic counterparts (see table 1 for more examples of vertex perfect graphs in general). There are however vertex perfect graphs that do not have analogs in number theory. For example, take the following graph shown with its perfect covering:

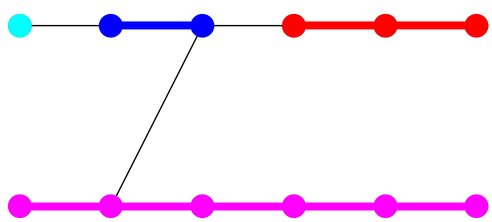

Figure 7. A vertex perfect graph on 12 vertices.

As a result of its edge structure, the graph of figure 7 has an order that is not a perfect number yet its vertex divisors are able to cover it. From this example we can intuitively grasp that VPGs might mimic the behavior of perfect numbers in some contexts while manifesting different behaviors based on the varying structures of connected graphs in other contexts. Investigating both these differences and similarities between vertex perfect graphs and perfect numbers is the subject of the rest of this paper.

1.2 Basic Properties of Vertex Perfect Graphs

What properties do VPGs have with respect to their vertex divisors? In this section, we begin to answer this question by proving theorems on connected graphs in general (of which VPGs are a subset) and by proving theorems specifically limited to VPGs.

As can be intuitively imagined, every connected graph is just the sum of many individual vertices joined together by edges. The following theorem formalizes this idea by stating that K_1 (the connected graph of order 1, or a graph on a single vertex) is the divisor of any graph excluding itself.

Theorem 1: K_1 is a divisor of all connected graphs except for itself.

Proof: Take any connected graph G such that $|G| \geq 2$ and erase all of its edges. What remains are $|G|$ copies of K_1. Hence, K_1 covers G.

Now suppose G is a connected graph such that $|G| = 1$. G must be K_1. Clearly we cannot remove some number of edges from K_1 to produce $n \geq 2$ copies of a new, connected graph. It follows that K_1 is not a vertex divisor of itself. ■

Because K_1 is a divisor of any graph, we can expect to find it in the perfect covering of every VPG. Furthermore, because we usually do not know much about the edge structure of graphs when trying to prove general results, K_1 is usually the *only* guaranteed divisor of a graph.

The next result relates the order of a connected graph to the orders of its divisors in a way similar to how positive integer relate to their proper divisors. Here is the theorem:

Theorem 2: If H is a vertex divisor of G, then $|H|$ properly divides $|G|$.

Proof: Suppose H is a divisor of G. Some of the edges of G can then be removed to produce a new graph consisting of $n \geq 2$ copies of H. The sum of the orders of the n copies of H must add to $|G|$ since no vertices are removed from G in this process. We then have $n|H| = |G|$, or $n = \frac{|G|}{|H|}$, which implies that $|H|$ must properly divide $|G|$. ■

The following corollary gives an upper bound on the order of a vertex divisor of a graph.

Corollary (2): If H is a vertex divisor of G, then $2|H| \leq |G|$.

Proof: Suppose that H is a divisor of G such that $2|H| > |G|$ and $n \geq 2$ is the number of copies of H that cover G. We then have

$$n = \frac{|G|}{|H|} < \frac{2|H|}{|H|} = 2,$$

which is a contradiction since $n \geq 2$. ■

The next theorem fittingly rounds out our knowledge of what can be known about a vertex perfect graph without any specific knowledge about it.

Theorem 3: If G is vertex perfect, then the orders of its divisors must add to $|G|$.

Proof: Suppose G is vertex perfect with order $|G|$. Let $N_1, N_2, \ldots, N_q$ be the various vertex divisors of G with corresponding orders $|N_1|, |N_2|, \ldots, |N_q|$. Because G is vertex perfect, its edges can be removed such that q components remain, each being a different divisor of G, (i.e. G has a perfect covering). Because no vertices are removed from the original copy of G when producing its perfect covering, there are $|G|$ vertices remaining in this new graph. We can conclude that the sum of the orders of each component is $|G|$. ■

To summarize our list of results thus far, if G is vertex perfect, then it has K_1 as a divisor, the orders of its divisors divide $|G|$, and those orders sum to $|G|$. The next section will make use of these three pieces of information to produce a powerful tool that “screens” graph orders for VPGs.

We now close this section with a final result concerning vertex divisors. It states that if some graph is a divisor of another graph, then any spanning graph of that divisor must also be a divisor of that graph. This highlights an important difference between positive integers and connected graphs: the latter can have multiple divisors of the same order.

Theorem 4: If $H_1$is a divisor of G and H_2 is a spanning subgraph of H_1, then H_2 is a divisor of G.

Proof: Let H_1 be a divisor of G and H_2 be a spanning subgraph of H_1. Because some $n \geq 2$ copies of H_1 can be produced from G by only deleting G’s edges and H_2 can also be produced from H_1 by also deleting some number of H_1’s edges, it follows that H_2 also covers G. ■

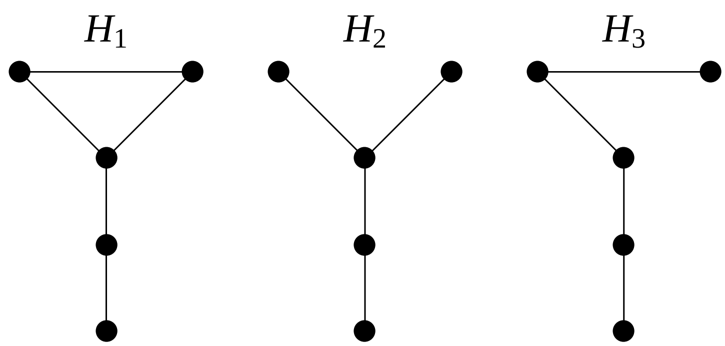

Figure 8. If H_1 is a divisor of a graph G, then H_2 and H_3 are also divisors of G.

1.3 Tools for Searching for Vertex Perfect Graphs

When looking for an object of interest, it is sometimes useful to know where not to look. The theorems developed in this section provide this information with respect to the orders of VPGs and also give information about the make up of their vertex divisors. The next and most important result of this category is informally known as **the Divisor Sieve**, which we now present for the reader.

Theorem 5 (The Divisor Sieve): Let G be a vertex perfect graph. If $q_1, q_2 \dots, q_n$ are all of the proper divisors of $|G|$ (besides 1) and m_i is the number of different graphs of order q_i that are vertex divisors of G, then the following equation must hold:

$$1 + m_1 q_1 + m_2 q_2 + \cdots + m_n q_n = |G|.$$

Proof: Let G be vertex perfect. By Theorem 3, the sum of the orders of G's divisors must add to $|G|$. By Theorem 2, the orders of G's divisors must be proper divisors of $|G|$. By Theorem 1, K_1 is guaranteed as a divisor of G. Hence, the above equation holds. ■

To illustrate the usefulness of the divisor sieve, we begin by providing an example of how it can be used to show that there are no VPGs of a specific order. By the contrapositive of the divisor sieve, if the above equation does not hold, then a graph G is not vertex perfect. Also, the values that each m_i can take on ranges from 0 to the minimum of $|G|/q_i - 1$ and the number of different connected graphs of order q_i (we leave it to the reader to verify this claim). Combining both of these facts allows us to demonstrate the following example that states there are no VPG's with order 14.

Example 6: If $|G| = 14$, then G is not vertex perfect.

By theorem 5, if a graph G with $|G| = 14$ is vertex perfect, the following equation must hold:

$$1 + 2n + 7m = 14,$$

where n is the number of divisors of order 2 and m is the number of divisors of order 7. This equation is only solved for the nonnegative integers n and m when $n = 3$ and $m = 1$. Because there is only one connected graph on two vertices (P_2), G is not vertex perfect. ■

The Divisor Sieve can also be used in a more general sense to show that there are an infinitely many orders for which there are no corresponding VPGs.

Theorem 7: If $|G|$ is prime, then G is not vertex perfect.

Proof: Let $|G|$ be prime. Since all positive integers less than $|G|$ are relatively prime to $|G|$ besides one, the divisor sieve gives $1 = |G|$, which will never be true. ■

Theorem 7 can be generalized into a theorem that states there are no VPGs with orders that are powers of primes. Additionally, we can also mark off graphs with orders that are certain kinds of products of primes from being vertex perfect. These two theorems are as follows:

Theorem 8: Let p a prime number. If $|G| = p^n$ for integer n, then G is not vertex perfect.

Proof: Let G be a graph with $|G| = p^n$ and suppose that G is vertex perfect. By the Divisor Sieve, the following equation must hold

$$1 + m_1p^1 + m_2p^2 + \cdots + m_{n-1}p^{n-1} = p^n$$

We can see that all of the above terms in the sum on the left side of the equation are divisible by p except for 1, which is a contradiction since the right side of the above equation is divisible by p. ■

Theorem 9: Let G be a graph with order that is the product of two primes p, q such that $p < q$. Let c_p be the total number of non-isomorphic, connected graphs on p vertices. If $1 + pc_p < q$, then G is not vertex perfect.

Proof: Let G be a graph with $|G| = pq$ and let $1 + pc_p < q$. If G is vertex perfect, then it must satisfy the divisor sieve, which yields

$$1 + pm_p + qm_q = pq$$

where m_p and m_q are the number of different graphs of order p and q respectively that are vertex divisors of G. We know that $c_p \geq m_p$, which implies that $1 + pc_p \geq 1 + pm_p$. Also, because qm_q must be smaller than $pq - 1$, m_q must be no greater than $p - 1$, which gives

$$m_q \leq p - 1$$

$$1 \leq p - m_q$$

$$q \leq q(p - m_q)$$

We can then take all of these results to produce the following contradiction:

$$1 + pc_p \geq 1 + pm_p = pq - qm_q \geq q \quad \blacksquare$$

To illustrate the usefulness of theorem 9, see the following example that demonstrates that there are no vertex perfect graphs of order 26:

Example 10: There are no vertex perfect graphs of order 26.

Proof: The number 26 is the product of 2 and 13 and there is only one connected graph on 2 vertices. $1 + 2 < 13$, so by theorem 9, there are no vertex perfect graphs of order 26. ■

The Divisor Sieve can also produces positive results; It can find all of the possible combinations of vertex divisors that a VPG can have for a specific order. For example, if G is a graph with order 21, it must satisfy the divisor sieve, which gives the following equation:

$$1 + 3m_2 + 7m_7 = 21,$$

where m_2 is the number of divisors of order 3 and m_7 is the number of divisors of order 7. The only solution to the above equation for m_2 and m_7 on their respective, allowed intervals is 2 and 2. This implies that if a VPG with order 21 exists, its set of divisors must include K_1, two graphs of order 3 and two graphs of order 7.

As an aside, it is important to note that just because there is a solution to the Divisor Sieve for a specific order, there may not necessarily exist a vertex perfect graph that corresponds to that order.

We now close this section with another tool for narrowing down the list of divisors of vertex perfect graphs. Its use is much more limited than that of the previously discussed results, but it gives a defining trait of the divisors of graphs with respect to their degree sequences.

Theorem 11: If $\delta(H) > \delta(G)$ or if $\Delta(H) > \Delta(G)$, then H is not a divisor of G.

Proof: Suppose $\delta(H) > \delta(G)$. If H is a divisor G, then the degree sequence of H must be exactly the same as each of the components of H produced from G by edge deletion. Because edges deletion will not cause the terms of G's degree sequence to increase, we can see that this will never be true.

Similarly, if $\Delta(H) > \Delta(G)$, we can see that edge deletion performed on G cannot form a component with greater degree than $\Delta(G)$. Hence, H cannot be a divisor of G. ■

Chapter 2
Perfect Paths and Graph Generation

2.1 Perfect Paths and Other Familiar Graphs

In this section, we prove theorems that state when certain types of graphs are vertex perfect. The most important of these concern paths. Knowing when paths are vertex perfect is useful for showing when other types of graphs are vertex perfect. Additionally, results on paths are useful in demonstrating other properties of vertex perfect graphs featured later in this chapter. We now begin by proving a lemma necessary our classification of when paths are vertex perfect.

Lemma 12: The vertex divisors of P_n are all paths with orders that properly divide n.

Proof: Let P_n be a path on $n \geq 2$ vertices. Deleting any edge in P_n produces a new graph on two, unconnected paths. This implies that the only graphs that can produced from paths by edge deletion are collections of unconnected paths. It follows from this fact and theorem 2 that the only graphs that can possibly be vertex divisors of P_n are paths whose order properly divides n.

Now let m be a proper divisor of *n.* It is trivial to demonstrate that P_n can be divided into $^n/_m$ components of P_m by edge deletion. We can thus conclude that the vertex divisors of P_n must then be all paths whose order divides *n.* ■

With help from the above lemma, we can demonstrate a direct link between perfect numbers and perfect paths.

Theorem 13 (The Perfect Path Theorem): P_n is vertex perfect if and only if n is perfect.

Proof: Let P_n be a path of order n. Two cases arise: n is either perfect or not perfect. If n is not perfect, then by Lemma 1, the sum of the order of the divisors must be greater or less than n, implying that P_n does not satisfy the divisor sieve and cannot be vertex perfect. If n is perfect, then by Lemma 1, the sum of the orders of the divisors of P_n is n and the divisors of P_n are all paths whose order properly divides n. If we choose one of the "end vertices" in P_n and let P_1 cover the first vertex of P_n, let the next m vertices where m is the second smallest proper divisor of n be covered by P_m, and repeat this process until we have exhausted all of the divisors of P_n, we can see that P_n is covered by its vertex divisors. This implies that P_n has a perfect covering when n is a perfect number, which completes the proof. ■

We can also prove a similar result for cycles and paths. This is because both the structure of cycles and wheel graphs are very similar to that of a path. The proof of these statements are featured below:

Theorem 14: The cycle graph C_n is vertex perfect if and only if n is perfect.

Proof: Assume C_n is vertex perfect. If we delete any one edge in C_n, the resulting graph is P_n. This implies that C_n shares all of the divisors of P_n, (all paths with orders that divide n). Because the only possible divisors of C_n are paths whose order divides n, the divisors of C_n and P_n are the same. It follows that C_n will only be vertex perfect when P_n is vertex perfect, implying that n is a perfect number when C_n is vertex perfect.

Now assume that n is perfect. As shown earlier, the divisors of C_n are all paths with order that divide n. Clearly any set of paths whose orders sum to n will cover C_n, so C_n must be vertex perfect if n is perfect. ■

Theorem 15: The wheel graph W_n is vertex perfect if and only if n is perfect.

Proof: Let W_n be the wheel graph pictured in the following figure:

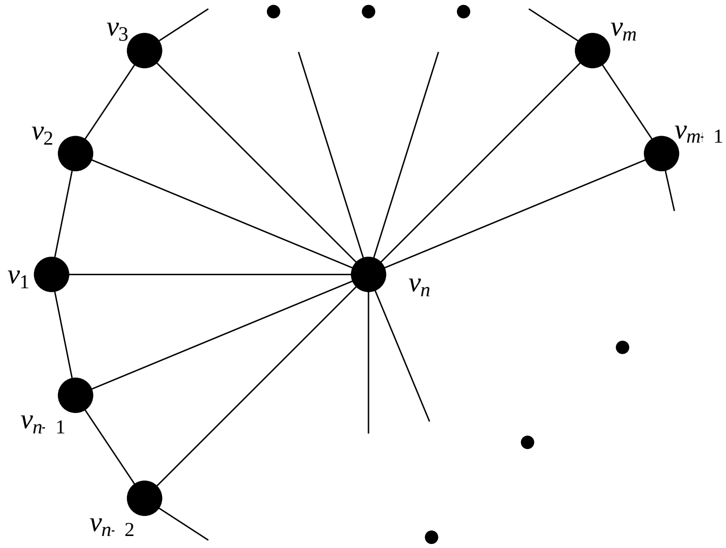

Figure 9. A general wheel graph.

Let H be a divisor of W_n with $|H| > 1$. Choose the center vertex v_n to be in one of the $\frac{n}{|H|}$ copies of H that W_n can be divided into by edge deletion. Two cases arise: either $|H| = 2$ or $|H| > 2$. If $|H| = 2$, then H is P_2. If $|H| > 2$, then the copy of H that contains v_n must also contain at least two other vertices in the "outer wheel" of W_n. This copy cannot contain all of the vertices on the "wheel" if $|H|$ is to properly divide n. This implies that if we use edge deletion to separate just the v_n containing copy of H from W_n, for some two vertices on the wheel of W_n, WLOG v_1 and v_m, the remaining v_2 through v_{m-1} vertices are then separated into a new component. This new component is a path on $m - 2$ vertices. Because the divisors of paths are only other paths, H must then be a path, which suggests that in either of the two mentioned previous cases, the divisors of W_n must all be paths. If we delete all of the edges incident with v_n except for the edge shared between v_n and v_1 in addition to the edge shared by v_{n-1} and v_1, we can see that W_n has P_n as a subgraph and that the divisors of W_n must be all paths whose order divides n. W_n will then only be vertex perfect when n is perfect, since this is the only instance for which the order of the divisors of W_n will sum to n. If n is perfect, then all of the orders of the divisors of W_n sum to n. Producing the corresponding perfect covering of W_n is trivial. ■

This gives three different theorems that state when certain related graphs are vertex perfect. The next and final theorem of this section is a negative result about another familiar class of graphs and when they are vertex perfect.

Theorem 16: Complete Graphs are not vertex perfect.

Proof: Let K_n be a complete graph with $|K_n| = n$. Because it is complete, every vertex of K_n must be adjacent to every other vertex. This implies that the divisors of K_n include any connected graph of order that divides n. Two cases arise; either the highest order divisor of K_n is at least four or is less than four. If the order of the **highest order divisor** (HOD) of K_n is less than four, then the HOD has order that is either one, two, or three. If the order of the HOD is one, then n is prime and K_n is not vertex perfect. If the order of the HOD is two, then either n is two or four, which gives K_n with an order that is prime or a power of a prime, implying K_n is again not vertex perfect. If the order of the HOD is three, then n is either three, six, or nine. If n is three or nine, then K_n has a prime or power of a prime order and K_n is not vertex perfect. If n equals six, then K_n is K_6. A quick check of the divisor sieve shows that K_6 is not vertex perfect.

Suppose the highest order divisor of K_n is at least four. For any graph of order four or more, the number of non-isomorphic, connected graphs on m vertices is greater than m.

Let q be the order of the HOD of K_n. Because n is not prime, q must be greater than one. For some positive integer p no greater than q, we must have that $pq = n$ since q properly divides n. We then have that

$$1 + qq \geq 1 + pq = 1 + n > n$$

If follows from the divisor sieve that K_n must not be vertex perfect. ■

2.2 From Perfect Paths to Infinity

If we know that a certain graph is vertex perfect, can we then use that graph to create a new graph that is also vertex perfect? For specific types of graphs, it turns out that the answer is yes. What follows is our first result found towards this question and it demonstrates that perfect paths with order n can be used to generate at least one additional vertex perfect graph of order $2n$.

Theorem 17: If n is perfect, then there exists a VPG with order $2n$.

Proof: Let P_n be a vertex perfect path. Select one of the vertices at either end of P_n and label it $v_{1,1}$. Label the vertex adjacent to $v_{1,1}$ as $v_{1,2}$, the vertex adjacent to $v_{1,2}$ as $v_{1,3}$, and so on until we reach $v_{1,n}$. If we add a new, separate component of P_n to this graph, label it similarly (with the exception that the first index is a two instead of a one), and add an edge between the vertices $v_{1,n-1}$ and $v_{2,n-2}$, we have the following graph J on two joined copies of P_n:

J

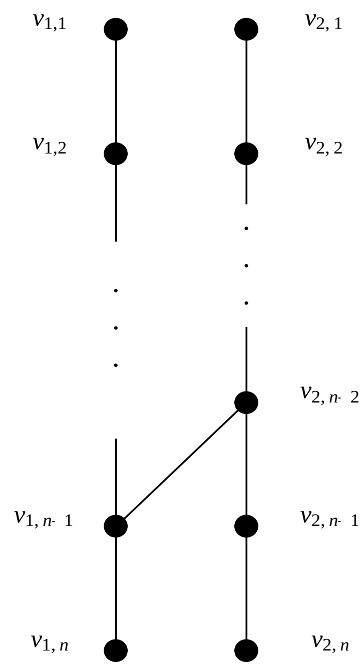

Figure 10. The graph *J* formed from joining two perfect paths of the same order.

J retains the divisors of P_n in addition to having P_n itself as a divisor. If these are the only divisors of *J*, then *J* must be vertex perfect.

By *J*'s construction, we have that $|J| = 2n$. Let $q_1, q_2 \dots q_m$ be all of the positive integers that properly divide *n*. Clearly these numbers also divide $2n$. The only other orders that can possibly divide $2n$ are $2q_1, 2q_2 \dots 2q_m$. For some *i* and *j* such that $1 \leq i, j \leq m$, it could be that $2q_i = q_j$. Let r_w be any $2q_i$ such that $2q_i \neq q_j$. To show that *J* is vertex perfect, we must show that there are no graphs of order r_w that are vertex divisors of *J* and that the only graphs of orders $q_1, q_2, \dots q_m$, n, that are divisors of *J* are the paths $P_{q_1}, P_{q_2} \dots P_{q_m}, P_n$.

We begin by showing that there are no graphs of order r_w that cover *J*. We know by our earlier reasoning and by the corollary to theorem 2 that $r_w < n$. If we utilize $v_{1,1}$ in a grouping of some graph of order r_w, $v_{1,2}$ must then also be used so as not to disconnect $v_{1,1}$ from the grouping, and the same for $v_{1,3}$ and on to v_{1,r_w}. Hence, if any graph of order r_w is to cover *J*, it must be P_{r_w} (this same argument can be applied to the orders of $q_1, q_2 \dots q_m$, saving us the trouble of tackling these cases later). By the definition of r_w, there is no positive integer t such that $t(r_w) = n$ (otherwise r_w is equal to some q_i). This implies that if P_{r_w} is to be a divisor *J*, one of the copies of it produced in *J* by edge deletion must utilize vertices from both the first and second copies of P_n. Furthermore, only one of the copies of P_{r_w} can utilize vertices in both copies of P_n since there is only one edge between both copies.

Examine the copy of P_{r_w} that is to contain $v_{1,n}$. Clearly this copy of P_{r_w} must also contain $v_{1,n-1}$. If $r_w = 2$, then P_{r_w} is not a divisor of *J* since it does not utilize vertices in both copies of P_n. So it must be that if this copy of P_{r_w} includes $v_{1,n-1}$, it must also include $v_{2,n-2}$. We must then have another vertex in this copy of P_{r_w} since, by definition, r_w must be even. To maintain the path structure of this copy of P_{r_w}, we must include $v_{2,n-1}$ or $v_{2,n-3}$, but we cannot include both. If we include $v_{2,n-3}$, then $v_{2,n-1}$ and $v_{2,n}$ cannot be joined into a path of order greater than two, so it must be that $v_{2,n-1}$ and $v_{2,n}$ must be the remaining vertices of this path. This implies that $r_w = 5$, which cannot be true since r_w is even. Hence, there are no graphs of order r_w that are divisors of *J* and the only graphs of order less than *n* that are divisors of *J* are paths that cover P_n.

It now remains to show that P_n is the only graph of order *n* that is a divisor of *J*. If we choose $v_{1,n}$ to be in a copy of some graph of order *n* that is to be a divisor of *J*, we must then select $v_{1,n-1}$. If we do not then select all of the remaining vertices in the first copy of P_n, some number of vertices less than n will not be able to be joined into a copy of a graph of order n. This implies that the only graph that can be a divisor of J of order n is P_n. ■

Of course the method of the above theorem is not the only way to generate a new vertex perfect graph from a specific path. The above structure is useful however in proving other results, namely that there are infinitely many vertex perfect graphs (a result not yet known for perfect numbers!). By strategically connecting J to other copies of J, we can generate an infinite number of vertex perfect graphs of order $2n(2^m)$ for nonnegative integer m.

Theorem 18: There are infinitely many vertex perfect graphs.

Proof: Take two copies of the same perfect path P_n and add an edge between $v_{1,n-1}$ and $v_{2,n-2}$. We then have the following graph J on $2n$ vertices that must be vertex perfect by the above theorem. Let G be the graph constructed in the following way: take some 2^x (x is a nonnegative integer) of labeled copies of J (i.e. $J_1, J_2, \ldots J_{2^x}$) and join the odd copies of J_m's $v_{2,n-2}$ to J_{m+1}'s $v_{1,n-1}$ and also join the even J_m's $v_{2,n-2}$ to J_{m+1}'s $v_{1,n-1}$ (provided that the next adjacent graph exists). G is the graph pictured below:

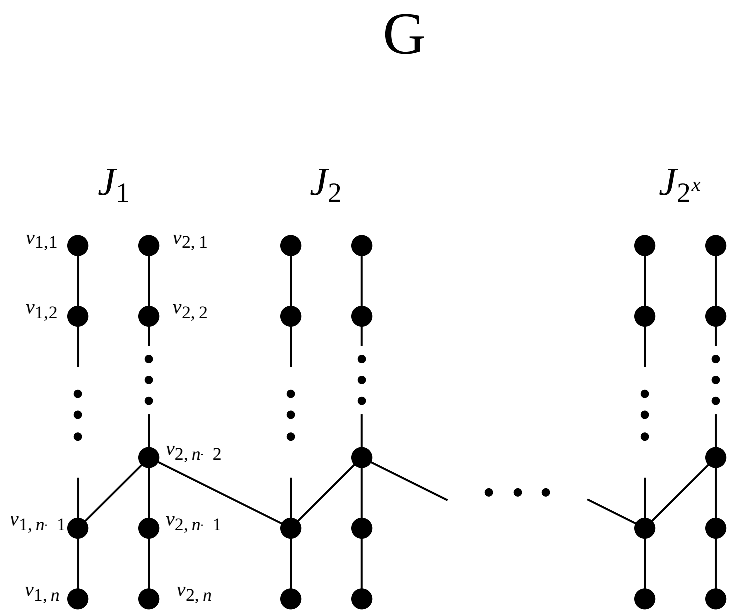

Figure 11. The graph G formed from joining copies of J together.

We wish to show that G is vertex perfect. To do this, we will show that G satisfies the following four criteria:

1. The only graphs of orders less than or equal to n that are divisors of G are P_n and P_{q_i}, where q_i is some proper divisor of n.
2. The only graphs of order $2^y(2n) < 2^x(2n)$ that are divisors of G are J and graphs consisting of full, adjacent copies of J.
3. No graphs of order $2^r(q_i) \neq q_j, n, 2^y(2n)$ (which are the only other possible orders divisors of G based on how G has been constructed) are divisors of G.
4. The graphs that are shown to be vertex divisors of G cover G.

We begin by demonstrating criteria one. Let q_i be a divisor of n. Suppose a graph H of order q_i is a divisor of G. We must then be able to erase edges from G to produce some number of isomorphic copies of H. If we include $v_{1,1}$ from J_1 in one of these copies of H, we must then include the next number of $q_i - 1$ vertices along this first copy of P_n in J_1 in the same copy of H. Since q_i is a proper divisor of n, by the corollary of theorem 2, we must have that $q_i \leq \frac{n}{2}$. Because the smallest perfect path is on six vertices, clearly the only way that $v_{1,1}$ and the next $q_i - 1$ can be included in a connected graph is as a path on q_i vertices. So the only graphs on order q_i that can be vertex divisors of G are P_{q_i}.

Now suppose a graph of order n covers G. It is obvious that P_n is a divisor of G. A graph H of order n other than P_n cannot be a divisor of G because if we include $v_{1,1}$ from J_1 to be in one of the copies of H that G would need to be able to be divided into, we must include all vertices up to at least $v_{1,n-1}$. If $v_{1,n}$ is then not included in the same component, it will be separated from the rest of the components since the inclusion $v_{1,n-1}$ in a different component separates it from the rest of the graph. We have thus satisfied criteria one.

We now move to demonstrate criteria two. Suppose a graph H of order $|H| = 2^y(2n) < 2^x(2n)$ is a divisor of G. It must be that vertex $v_{1,1}$ of J_1must be in one of the components of H that G can be divided into by edge deletion. If we include $v_{1,1}$ of J_1, then we must include J_1. This is because J_1 is a tree, which implies that not including a vertex in J_1 will separate that vertex into a component of order less than $|H|$. Furthermore, if we include any vertex from a component of J_u, we must then include all of its vertices by the same reasoning. It follows that the only graphs of $|H|$ that are divisors of G are graphs consisting of full, adjacent copies of J. Also, H will contain some power of two copies of J's (less than the total number of J's) since these are the only subgraph of order $2^y(2n)$ that properly divide $|G|$.

We now move to the third criteria. Take the number $2^r(q_i)$. By its definition, for some number $m \leq x$, we must have that $2^{m-1}(2n) < 2^r(q_i) < 2^m(2n)$. If a graph H

with order $|H| = 2^r(q_i)$ is to be a divisor of G and we include $v_{1,1}$ in one of the copies of H that G can be divided into by edge deletion, if follows that we must include the first full 2^{m-1} copies of J in the same component (not doing so would separate some number of vertices less than $2^r(q_i)$ from G). By the same reasoning, because within each copy of J, each copy of P_n has only one vertex that is adjacent to vertices in other copies of P_n, H must either include some integer number of copies of full, adjacent J's or must be some integer number of copies of full adjacent J's plus half of one copy of J. A graph of either construction will have an order that will not properly divide $2^x(2n)$, so there are no graphs of order $2^r(q_i)$ that are divisors of G.

For the final criteria, we need to show we can assemble the same number of copies of J that are in G from the divisors of G. We can assemble one copy of J from P_n and its divisors. The other divisors of G are full, adjacent $2^m < 2^x$ copies of J. Thus, it follows that we must show that the number of J's that can be assembled from the divisors of G sums to the number of J's present in G. The sum to be shown is as follows:

$$1 + 1 + 2 + 4 + 8 + \cdots + 2^{x-1} = 2^x,$$

which is clearly follows from the formula for a geometric series. ■

2.3 Progress on Generalizing Generation

As seen in the last section, it appears that new vertex perfect graphs can be generated from known vertex perfect graphs by minimally connecting identical copies of it with new, strategically placed edges. This approach to generating new vertex perfect graphs has yielded great success for specific types of graphs in which plenty is known about the corresponding edge structures. For vertex perfect graphs in general however, only a partial result has been able to be produced. We now define this idea of generation and afterwards state and prove the mentioned partial result.

If G is a vertex perfect graph, then we know that G is covered by its divisors. Suppose that G has n different spanning graphs. None of theses spanning graphs are divisors of G since their orders do not properly divide $|G|$. The orders of these spanning graphs do however properly divide any multiple of $|G|$ greater than $|G|$. If we sum the total number of vertices that are in the divisors of G (which is $|G|$ since G is vertex perfect) and in all of the n different spanning graphs of G, we get the number $(n+1)|G|$. It follows that if we can take $(n+1)$ copies of G and join them without producing any new vertex divisors besides the spanning graph of G, then that newly produced graph should be vertex perfect. This process is referred to as **perfect generation.** For a visual representation of this process, see the following figures:

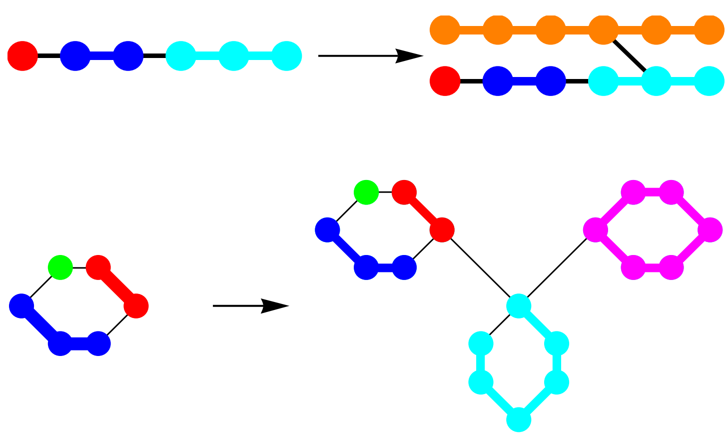

Figure 12. Examples of Perfect Generation.

We now present the aforementioned result.

Theorem 19: Let G be a vertex perfect graph and let n be the number of spanning graphs of G. If the graph J formed by joining the same vertex in n copies of G to a vertex v_j in a $(n + 1)$ copy of G does not have any divisors with order less than $|G|$ that properly divide $|J|$ but not $|G|$, then J is vertex perfect.

Proof: Let G be vertex perfect and let n be the number of spanning graphs of G. Suppose we can form the above described graph of J without producing any new divisors with order less than $|G|$ that properly divide $|J|$ but not $|G|$. J is then the graph featured in the following figure:

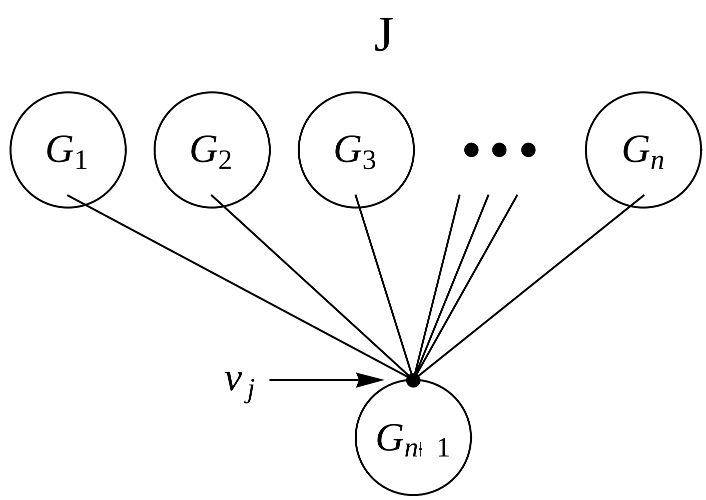

Figure 13. The Generated Graph J.

We wish to show that J is vertex perfect. Here are the criteria that must be satisfied:

1. We must show that the original divisors and spanning graphs of G are divisors of J and that they cover J.
2. We must show that there are no other new graphs of the orders that divides$|G|$ that are divisors of J.
3. We must show that there are no divisors of J of an order that properly divide $|J|$ but not $|G|$.

The first criterion follows from the fact that J has been constructed from full copies of G. Each of the copies of G can be broken into its original divisors and clearly each of the spanning graphs of G is a new divisor of G. G can be covered by this set of divisors by allocating each spanning graph to n of the $n + 1$ copies of G and reserving a final copy of G to break into its original divisors.

We now move to the second criterion. Suppose a new graph H of order that divides $|G|$ (not necessarily properly) is a divisor of J. If we try to divide J into $|J|/|H|$ copies of H by edge deletion, we must select the vertex v_j to be in one of the newly produced copies of H. If we select v_j and then any number of vertices in another copy of G besides G_{n+1}, the remaining vertices of G_{n+1} will be broken off into a new component with a number of vertices that that is not divisible by $|H|$. This implies that H is a divisor of J if and only if it is a divisor of G. The second criterion is then satisfied.

We now move to the final criterion. We break into two cases: first suppose that J has a new divisor H of order that is greater than or equal to $|G|$ and suppose H has the other property mentioned in criterion 3. If we select the vertex v_j to be in one of the copies of H that J can be divided into by edges deletion and select any number of vertices in G_{n+1} to be in that same covering, we must then select all of the vertices in G_{n+1} to be in this same covering otherwise some number of vertices less than $|H|$ will be broken off into a separate component. By this logic however, all vertices of J must be included in the same copy of H that includes v_j. Hence there are no new divisors of J with order greater than $|G|$ with the specifications of criterion 3.

H cannot have order less than $|G|$ and have the other property of criterion 3 by the assumptions of this theorem. This completes the proof. ■

2.4 The Vertex Perfect Graph Catalogue

Using previously listed theorems, we now produce a catalogue of orders 1 to 100 and whether or not exists a corresponding vertex perfect graph.

By theorem 7, there are no vertex perfect graphs of prime order. This eliminates the following orders: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

By theorem 8, there are no vertex perfect graphs with orders that are powers of primes. The following orders are then also eliminated: 1, 4, 8, 9, 16, 25, 27, 32, 49, 64, and 81.

Theorem 9 allows us to eliminate any order that is the product of 2 and another prime greater than three, including: 10, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, and 94. By theorem 9, we can similarly eliminate orders that are the product of 3 and a prime greater than 7, including 33, 39, 51, 57, 69, 87, and 93. The following orders simply do not satisfy the Divisor Sieve: 15, and 50.

The orders 52 and 68 are shown in the following theorem not to contain any corresponding VPGs:

Theorem 20: There are no vertex perfect graphs of order 52 or 68.

Proof: Suppose G is a vertex perfect graph of order 52 or 68. If $|G| = 68$, then by the Divisor Sieve, G must have one of the two following combinations of divisors: four divisors of order 4, one of order 17 and one of order 34 or G must have only four divisors of order 4 and three of order 17. If $|G| = 52$, then by the divisor sieve, G

must have one of the two following combinations of divisors: three divisors of order 4, one of 13 and one of 26 or three divisors of order 4 and three of order 13. So G must have at least three divisors of order 4 and no divisors of order 2. As there are only 6 different connected graphs of order 4 with only one does have a divisor of order 2, by theorem 4, G must have a divisor of order 2. This is a contradiction. ■

From the methods used to prove theorem 18, we know that there will be vertex perfect graphs with orders: 6, 12, 24, 28, 48, 56, and 96. The following orders have been found to have vertex perfect graphs either by perfect generation or trial and error:

18, 20, 21, 30, 40, 45, 54, 55, 60, 72, 78, 80, 84, and 90. The sixteen remaining, unlisted orders may possibly have corresponding vertex perfect graphs. These results are summarized in the table on page 76.

Chapter 3
Vertex Perfect Trees

In this chapter, we begin an investigation of trees in the context of this generalization. Trees are relatively easier to research than other broader types of graphs since much is known about their structure. They serve as a nice "first step" for proving (or disproving) difficult conjectures. The main goal of this chapter is just that; we wish to uncover as many potent theorems about trees and vertex perfect trees so that it becomes possible to prove more interesting theorems in the limited case of these minimally connected graphs.

3.1 The Reducibility of Trees and the Trusty Bridge Theorem

How are trees related to their divisors? The answer to this question begins with the following lemma, which dictates the number of edges that must be deleted when producing a divisor covering of a tree.

Lemma 21: Let T be a tree and let Q be a vertex divisor of T. The number of edges that must be deleted to produce Q's covering onto T is $|T|/|Q| - 1$.

Proof: If T is a tree and Q is a vertex divisor of T, then by theorem (), Q's covering onto T consists of $|T|/|Q|$ copies of Q. As every edge in a tree is a bridge, the deletion of any edge in T increases the number of its components by one. Because T has only one component, deleting $|T|/|Q| - 1$ edges from T will produce $|T|/|Q|$ new components. ■

In addition to providing the leverage needed for the next result, this lemma demonstrates an interesting consequence of the minimal connectedness of trees and their divisor coverings. We know that deleting an edge from a graph either increases the number of its components by one or leaves the total number of its components unchanged. This allows us to conclude that the minimum number of edges that must be deleted to produce a divisor covering of a graph is one less than the quotient of their orders, as this would imply that a new component has been produced with each edge deleted in that graph. In short, the above lemma states that the number of edges to be deleted to produce a divisor covering for a minimally connected graph is also minimal.

We now use this lemma to prove the following important theorem concerning trees.

Theorem 22: Let T be a tree, Q a vertex divisor of T, and E the set of edges that must be deleted to produce T's Q covering. The graph formed by contracting all edges not in E is also a tree, and has order $|T|/|Q|$.

Proof: If we contract all of the edges of T not in E, then all of the vertices in each copy of Q of T's Q covering are "fused" into a single vertex. This follows from the fact that only edges incident with two vertices in the same copy of Q in T's Q covering will be collapsed. Edges incident with two vertices in different copies of Q must belong to E and are accordingly preserved in this new graph.

This implies that the graph produced from collapsing all of the edges not in E must have $|T|/|Q|$ vertices. From the previous lemma, we also know that this graph must have size $|T|/|Q| - 1$. It is known that connected graphs with size one less than their order are trees, so this resulting graph must also be a tree. ■

This result demonstrates that divisors of trees compose their parent graphs in a way that reduces to the structure of simpler trees. For example, the following graph T of order 24 has vertex divisors with order 4 and 12 (see the figure below). We can then see that the coverings of these vertex divisors map to simpler trees produced by collapsing the edges within each divisor as they occur in T. We call these resulting graphs **reducible trees**.

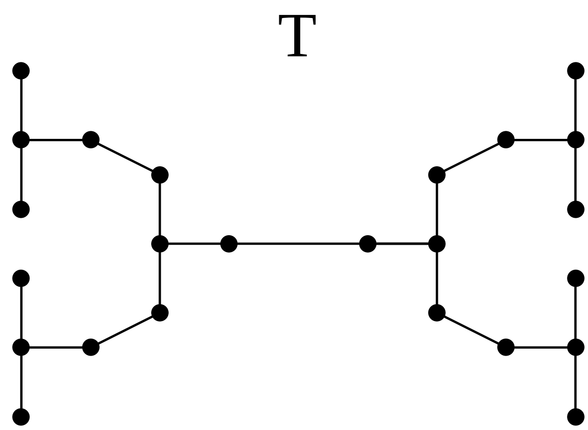

Figure 14 (a). A tree of order 24.

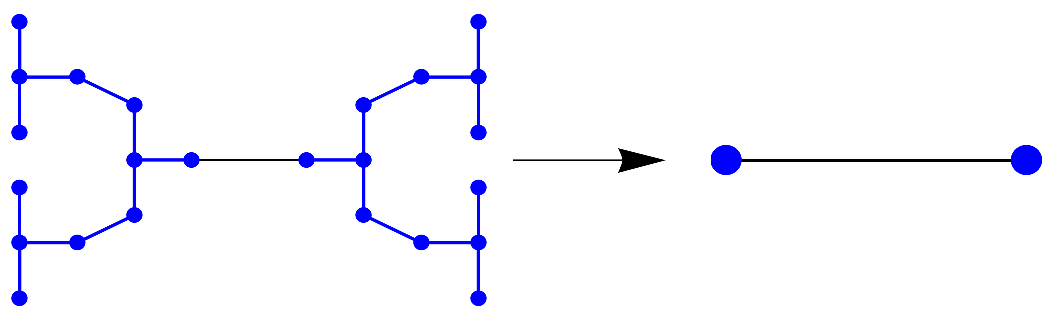

Figure 15 (a). An example of a divisor covering of a tree and its corresponding reducible tree.

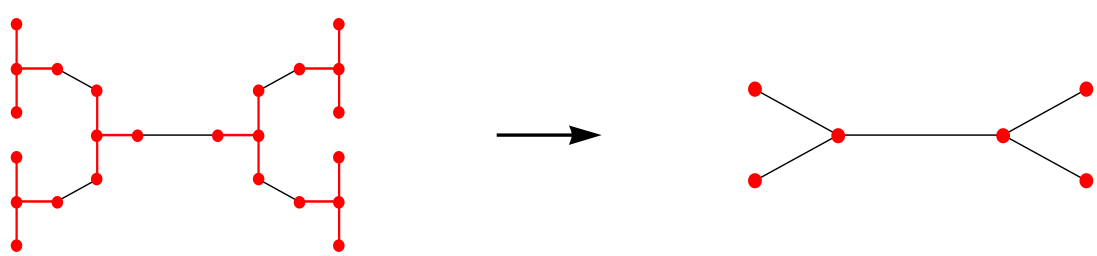

Figure 15 (b). Another example of a divisor covering of a tree and its corresponding reducible tree.

The previous theorem can be used to prove the existence of a certain type edge found in trees called a trusty bridge. A **trusty bridge** in a connected graph G with vertex divisor Q is an edge that when deleted, separates a copy of exactly Q from the rest of G. It is essentially the edge that connects a leaf in a reducible tree. As such, we call the copy of Q that can be separated from T by deleting its corresponding trusty bridge a **reducible leaf**.

The importance of trusty bridges existing in trees cannot be overstated. This will be demonstrated in the proofs that accompany theorems in later sections, as they all rely on the existence of trusty bridges (hence the name "trusty bridge"). We finish this section by stating this result.

Theorem 23 (the Trusty Bridge Theorem): Let T be a tree and let Q be a vertex divisor of T. There exist at least two trusty bridges in T that produce Q.

Proof: By the previous theorem, we know that the graph formed by collapsing all edges not in a Q covering of T is a tree. We also know that each vertex in that reducible tree corresponds to a copy of Q in T, and that the one edge that joins any two vertices in the reduced tree must correspond to a single edge that joins any two copies of Q in T. As proven on page 35 in [A1], every tree with order greater than two must have at least two leaves. These two leaves in the reduced tree correspond to two copies of Q that are only adjacent to one other copy of Q in T. The edge that provides this adjacency is a trusty bridge. ∎

3.2 Vertex Peninsulas and the Binary Divisor Sieve

It is often advantageous to utilize certain sections of a graph when demonstrating that it does (or does not) have a given vertex divisor. For example, if we want to show that the following graph does not have any vertex divisors of order 4, we could arbitrarily choose vertex v_1 to be in some divisor covering of order 4. We can see that choosing v_1 implies that we must also choose vertices v_2, v_3, and v_4 to be in the same vertex divisor, as not choosing all of them causes at least one vertex to be separated into its own component with order less than 4. Separating the $v_{1\text{-}4}$ component from the rest of the graph leaves C_8, a graph whose vertex divisors only include paths. Because the group of four labeled vertices do not form a path of order 4, this order 12 graph must not have any vertex divisors of order 4.

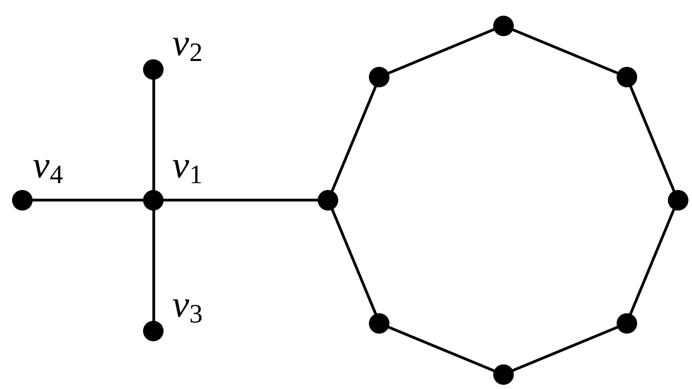

Figure 16.

The grouping of labeled vertices in the above figure restricts what the divisors of the entire graph can look like. We call this specific type of structure within a graph a **vertex peninsula** and define it as follows;

Let G be a connected graph, let v be a cut vertex, and let $\{H_1, H_2, \ldots, H_n\}$ be the set of components produced in G when v is deleted. Also, Let $S = \{H_i, H_j, \ldots, H_r\}$ be some arbitrary, non-trivial subset of the set $\{H_1, H_2, \ldots, H_n\}$. The subgraph in G

formed by the vertex v, the vertices of S, and all of their corresponding adjacencies form a vertex peninsula of order $1 + |H_i| + |H_j| + \cdots + |H_r|$. We call the cut vertex that connects the vertex peninsula to the rest of the graph the **connector**.

The following lemma formalizes our intuition of how vertex peninsulas affect the divisors of graphs.

Lemma 24 (The Vertex Peninsula): Let G be a connected graph. If G has a vertex peninsula P and a vertex divisor Q such that $|P| = |Q|$, then $Q \subseteq P$.

Proof: Suppose the hypothesis of this lemma. Select the connecter of P to be in one of the $|G|/|Q|$ copies of Q in G*'s* Q covering. If we do not select some number of the remaining vertices in P to be in the same copy of Q as the connecter of P, then those vertices will be broken into their own component(s) when the "connector-containing" copy of Q is separated from G. It follows that all of the vertices in P must be selected to be in this same component, implying that Q is some spanning graph of P. ■

This lemma is particularly important to trees. Before we can demonstrate this however, we need the following theorem that states the only vertex divisors of trees are trees.

Theorem 25: If T is a tree and Q is a vertex divisor of T, then Q is also a tree.

Proof: Suppose Q is not a tree. Q must then have at least one cycle, which would imply that T also has at least $|T|/|Q|$ cycles. This cannot be true, so Q must also be a tree. ■

Combining the last two results along with the trusty bridge theorem from the previous section allows us to prove a surprising fact about the divisors of trees. We can already see that the divisors of any given order of a tree are restricted by the fact that they must also be trees. This limits the number of divisors per order that any tree can have. It turns out that the divisors of trees are also limited to one per order; that is, if T is a tree and Q_1 and Q_2 are different vertex divisors of T, then $|Q_1| \neq |Q_2|$.

Theorem 26: Let T be a tree and Q a vertex divisor of T. Then Q is T's only vertex divisor of order $|Q|$.

Proof: If T is a tree and Q is one of its vertex divisors, then by the trusty bridge theorem, there exists a bridge in T that when deleted separates one copy of Q into its own component. Label this edge e, and examine the vertex that is both incident with e and in this specified copy of Q. Clearly this vertex is the connector of a vertex peninsula with order $|Q|$. So if T has any other vertex divisors of order $|Q|$, it must be a spanning subgraph of Q by the vertex peninsula lemma. By the previous theorem, Q must be a tree. Because the only spanning graph of a tree is itself, Q is the only vertex divisor of T with order $|Q|$. ∎

This theorem has implications for vertex perfect trees, especially for the divisor sieve that governs the possible orders they appear on. When an order that corresponds to a tree is put through the sieve, the constants corresponding to the number of vertex divisors a vertex perfect graph has of a given order are limited to one or zero. This new "binary version" of the divisor sieve is summarized in the following theorem.

Theorem 27 (The Binary Divisor Sieve): Let T be a vertex perfect tree and let $q_1, q_2, \ldots q_n$ be the proper divisors of $|T|$ that are greater than one. The following equation holds:

$$1 + c_1 q_1 + c_2 q_2 + \cdots + c_n q_n = |T|$$

where c_i is the number of vertex divisors that T has of order q_i and is either 0 or 1.

Proof: The above equation is just the divisor sieve with restricted values for the c_i terms. If T is vertex perfect, then the divisor sieve must hold. It remains to show that each c_i takes on the mentioned, restricted values. The restriction follows from the previous theorem. ■

In other words, the only orders eligible to have vertex perfect trees are those that are some subset of their proper divisors. These numbers are called **semi-perfect**, and the divisor sieve in this case simply finds these numbers and the subsets of their divisors that sum to them.

3.3 More Properties of Vertex Divisors

As the title suggests, in this section we provide a number of results that will help paint a generalized picture of trees relative to their divisors. We begin by establishing the following theorem about their divisor coverings:

Theorem 28: Let T be a tree and Q a vertex divisor of T. The divisor covering of Q onto T is unique.

Proof: If Q is a vertex divisor of T, then Q must be a tree. It also follows that T can be assembled by taking $|T|/|Q|$ copies of Q and adding edges amongst them. Let this set of edges added to produce T be called E. If we remove E, then we produce T's Q covering.

Suppose another covering of Q can be produced by removing edges not strictly in E. Because T is a tree, if we delete any one edge outside of E, we separate T into two components. Deleting an edge not in T must divide one of the earlier mentioned copies of Q, implying that the two produced components have orders that are some nonnegative multiple of $|Q|$ plus some positive number of vertices less than $|Q|$. Because neither of these components are divisible by $|Q|$, a covering of Q cannot be achieved in this case. Hence, the only covering of Q onto T is produced when only deleting edges from E. ■

So there is only one way to break a tree into copies of one of its divisors. It is important to note that this is not necessarily true of perfect coverings onto vertex perfect trees. For example, the following image shows three different ways to form a perfect covering of P_6.

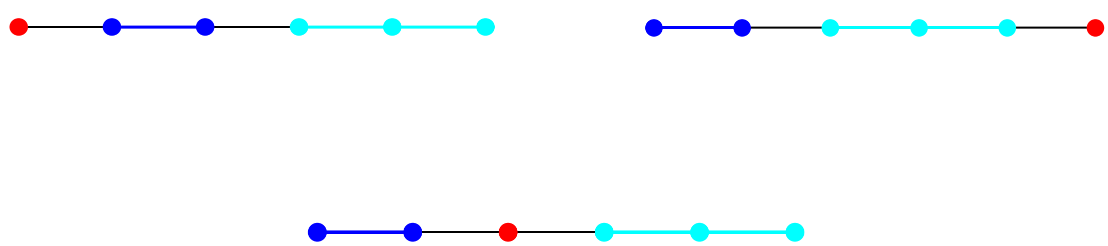

Figure 17.

The next two theorems demonstrate how the presence of multiple vertex divisors in a tree imply structure within its higher order divisors. In short, the presence of many divisors forces higher order divisors to assume a shape that accommodates lower order divisors. Here is the first theorem:

Theorem 29: Let T be a tree with vertex divisors Q_1 and Q_2. If $|Q_1| < |Q_2|$ then $Q_1 \subset Q_2$.

Proof: If Q_2 is a vertex divisor of T, then there exists a trusty bridge e in T that when deleted separates one copy of Q_2 into its own component. Let the vertex incident with e in this copy of Q_2 be called v. Choose v to be in one of the $|T|/|Q_1|$ copies of Q_1 that T can be divided into by edge deletion. Regardless of the number of vertices we choose to include from the Q_2 reducible leaf to be in the copy v containing Q_1 , because $|Q_1| < |Q_2|$, some number of vertices less than $|Q_2|$ will be separated into their own component when the v containing Q_1 is separated into its own component. It follows that if Q_1 is a divisor of T, then the mentioned component that is produced must have Q_1 as a divisor, implying that $Q_1 \subset Q_2$. ■

The above theorem is exciting in that it tells us much about what trees with multiple divisors must look like. As lower order divisors are subgraphs of their higher order divisors, higher order divisors are guaranteed to contain at least one full copy of their lower order divisors within their structure. Thus if we begin by examining the HOD of a tree, we can guarantee that a full copy of the next lowest order divisor is in this graph. Similarly, the next lowest order divisor must also contain a copy of the next lowest order divisor, and so on until we reach K_1. This provides a "core" of divisors within the HOD, the HOD which in turn has a unique covering onto its parent graph that also mimics the behavior of a tree. We summarize this picture with the following figure.

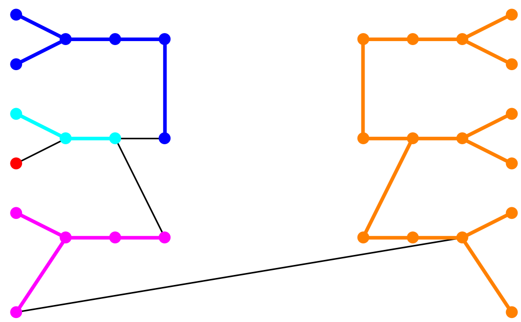

Figure 18.

The above tree of order thirty is vertex perfect and is shown with its perfect covering. Its highest order divisor is the orange subgraph on the right with order 15, featured in the next image.

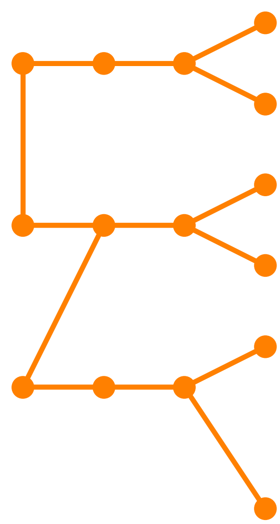

Figure 19.

We can then see that a full covering of the original VPT's next highest order divisor of order 6 within the order 15 HOD.

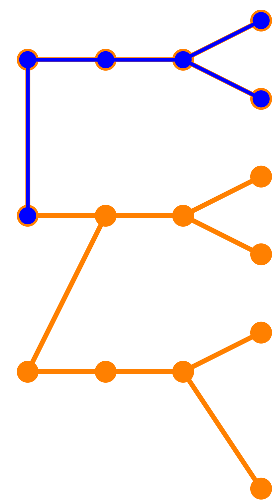

Figure 20

We then get a successive relationship of the next lower order divisors being contained with each of the higher order divisors:

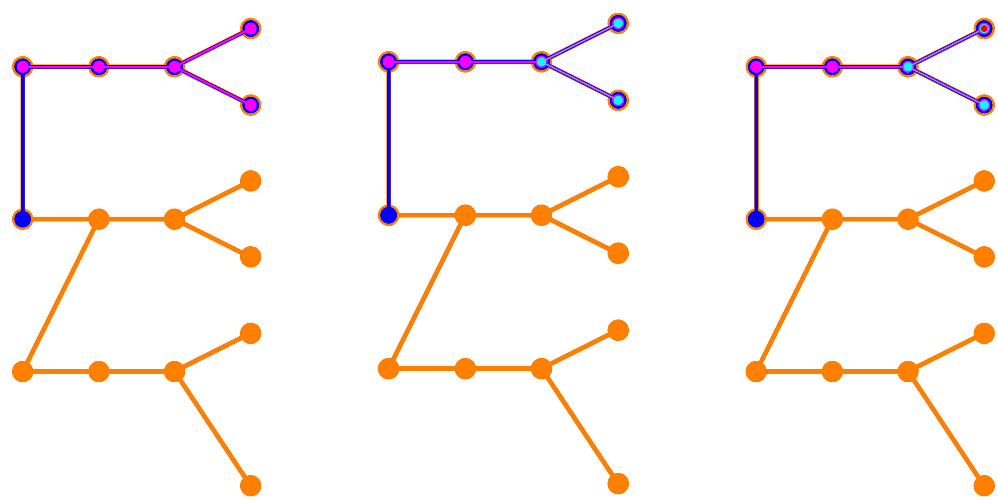

Figure 21.

More generally, the following figure demonstrates how we represent this idea of higher order divisors of a tree containing the lower order divisors.

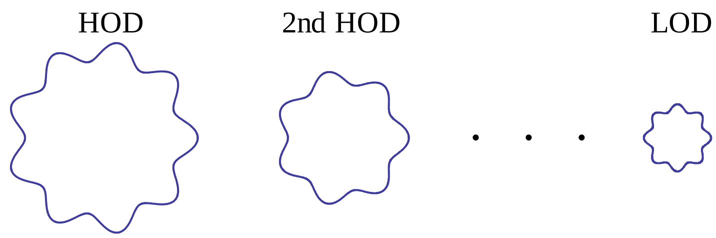

Figure 22 (a).

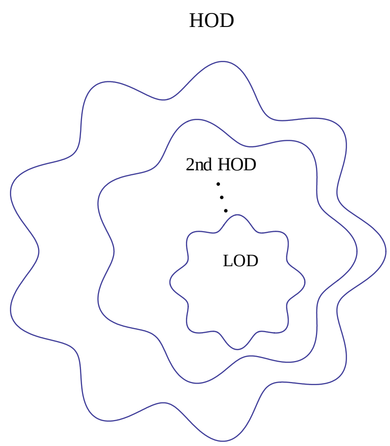

Figure 22 (b).

It turns out that we can extract even more information from the divisors of T; we can also figure out how many copies of a lower order divisor are contained in a higher order divisor. This is done in a surprisingly similar way to how the number of divisors in a graph is calculated: by simply taking the quotient of the orders of the divisors. Here is the statement and proof:

Theorem 30: Let T be a tree and let Q_1 and Q_2 be vertex divisors of T. WLOG, let $|Q_1| < |Q_2|$. The number of full copies of Q_1 contained in Q_2 is given be $\lfloor |Q_2|/|Q_1| \rfloor$. Furthermore, if $|Q_1|$ is a proper divisor of $|Q_2|$, then Q_1 is a vertex divisor of Q_2.

Proof: Assume the hypotheses of the above theorem. If Q_2 is a vertex divisor of T, then there exists a trusty bridge e that connects a Q_2 reducible leaf to the rest of T. Choose the vertex v incident with e that is in the Q_2 reducible leaf to be in one of the copies of Q_1 that T can be divided into by edge deletion.

Suppose that $|Q_1|$ does not divide $|Q_2|$. Then the copy of Q_1 containing v must contain a number of vertices from the Q_2 reducible leaf such that when it is separated from T, the remaining number of vertices in Q_2 is some multiple of $|Q_1|$. If the number of vertices is not a multiple of $|Q_1|$, then clearly the vertex covering of Q_1 onto T cannot be accomplished as the removal of the copy of Q_1 containing v separates those remaining vertices into their own component. In addition, those remaining vertices must each be able to be divided into copies of Q_1, or else Q_1 is not a vertex divisor of T. It follows that the order of Q_2 can be written as

$$|Q_2| = n|Q_1| + r$$

where n is multiple number of $|Q_1|$ left in Q_2 when the v containing Q_1 is deleted and r is the number of vertices that the v containing Q_1 claims from the Q_2 reducible leaf. Dividing both sides by $|Q_1|$ gives the following equation, which proves the theorem in question when Q_1 does not divide Q_2:

$$\frac{|Q_2|}{|Q_1|} = n + \frac{r}{|Q_1|}$$

Now suppose that $|Q_1|$ does divide $|Q_2|$. It follows that the copy of Q_1 containing v must have all of its vertices in the Q_2 reducible leaf. If this is not true, then producing the Q_1 covering causes some number of vertices in the Q_2 reducible leaf not divisible by $|Q_1|$ to be separated into their own component (following from the fact that v is incident with the trusty bridge e). This implies that all of the vertices of Q_2 must be able to be divided into some integer number of copies of Q_1. Hence Q_1 is a divisor of Q_2. From theorem 30, we know that the number of copies of a divisor in a parent graph is the quotient of their orders. ■

To illustrate the above theorem, we revisit the HOD of the previous figure 18. The HOD has order 15 and the highest order divisor has order 6, so we should expect to find at least 2 copies of the order 6 divisor in the HOD:

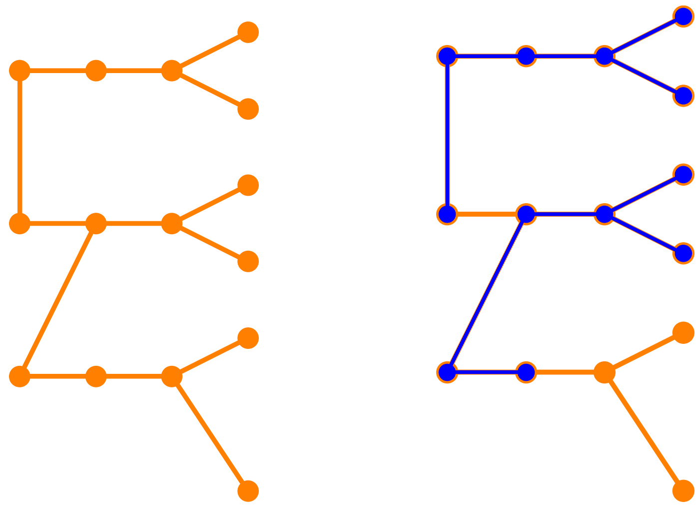

Figure 23.

3.4 The Interchangeable Parts Theorem

This section concludes our investigation of the properties of trees in this generalization. We close with a theorem that demonstrates how close trees and their divisors come to mimicking the behavior of integers in the realm of number theory. Because of its consequences, this theorem has been named the interchangeable parts theorem, and is stated as follows:

Theorem 31 (the Interchangeable Parts Theorem): Let T be a tree and let d_1 and d_2 be vertex divisors of T. T must have a vertex divisor of order $G = GCD(|d_1|, |d_2|)$.

Proof: The following proof is long and is filled with a number of partial results. As such, we have decided to organize some of these results by declaring them as lemmas within the proof. We do this so that during a first time read through this theorem, the reader can choose to skip the details associated with the partial results and instead get a feel for the overall structure of the larger theorem. Of course, the proofs of the lemmas remain where they normally would without this arbitrary organization, so if the reader would like to wade into all of the gritty details, all they need to do is read from top to bottom.

Assume the hypotheses of this proof. Without loss of genrality, let $|d_2| > |d_1|$. If $G = 1$, then the result is trivial as all graphs have a vertex divisor of order 1. If $G = |d_1|$, the result is again trivial. For the remainder of this proof, assume that $G \neq 1, |d_1|$.

Begin by deleting an edge in T that separates T into two components, both of which have orders that are divisible by G. Such an edge must exist since d_1 and d_2 are vertex divisors of T. In the newly produced components, continually repeat this process until what remains are a set of components all with orders that are some multiple of G and that can no longer have edges deleted in them to produce new components also with orders that are multiples of G. Call the set of edges deleted in this process E, and call these newly formed components constituent graphs (or just constituents for short). Let C be the complete set of constituents produced from T by deleting E (note that each element of C may not be unique by C's definition; that is, a given constituent may appear more than once in C).

Lemma: E and C are unique.

Sub-proof: We claim E must be unique, but suppose it is not. Let E_1 and E_2 be edge sets that satisfy the definition of E and suppose E_1 is not equivalent to E_2. Also, let C_1 be the set of constituents that produced when E_1 is deleted. If E_1 is a subset of E_2, then not all of the edges that can be deleted to separate T into smaller and smaller components with orders that are multiples of G are deleted when deleting E_1. A similar problem arises if E_2 is a subset of E_1, so if E_1 and E_2 are different, either E_2

must contain an edge from T not contained in E_1 or vice versa. Without loss of generality, suppose E_2 that contains at least one edge e that E_1 does not contain. e must be an edge that connects two vertices in the same constituent c_e from C_1. By definition, deleting e separates c_e into two components, each with order that is not a multiple of G. This implies that only deleting e in T produces two new components with orders that are not multiple of G since both of these newly produced components contain some non multiple of G vertices corresponding to the separated c_e plus some multiple of G vertices corresponding to the remaining, "intact" constituents. This implies that e cannot be in any E, and E must be unique.

If E is unique, then C is also unique. ■

The edges deleted to produce T's d_1 and d_2 coverings are each subsets of E, since deleting any edge in the d_1 covering produces components with orders that are multiples of $|d_1|$ and deleting any edge in a d_2 covering produces components that with orders that must be multiples of $|d_2|$. This implies that these various constituents as they appear in T are preserved when producing said coverings (that is, when producing the d_1 and d_2 coverings of T, we do not delete any edges in the constituents as they appear in T). It follows that both T's d_1 covering and T's d_2 covering can be produced by adding edges amongst the various elements of C.

We now break to produce some necessary information about the structure of T's d_1 covering relative to T's d_2 covering. By theorem 23, we know that if d_2 is a vertex divisor of T, then there must exist some d_2 trusty bridge in T that corresponds to a d_2 reducible leaf. If d_2 is the graph pictured to the left in the following figure, T is then the following graph of the right:

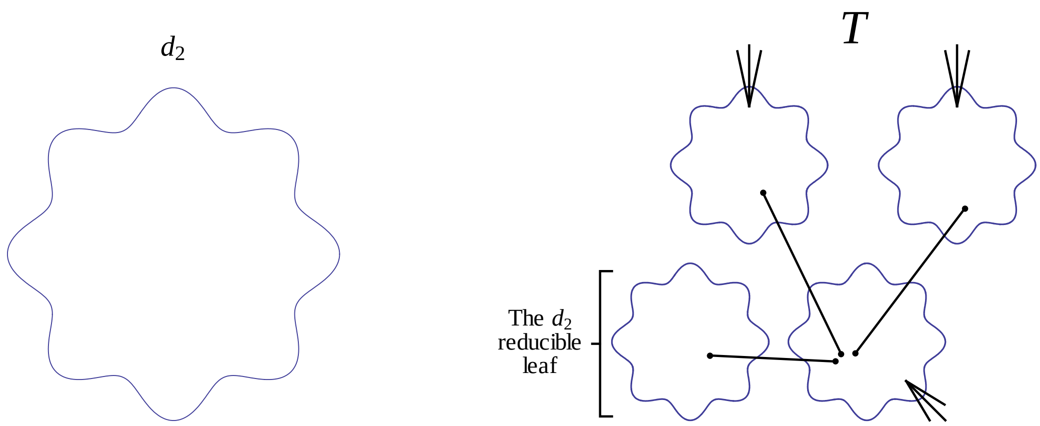

Figure 24.

By theorem 30, we also know that there are $n = \lfloor |d_2|/|d_1| \rfloor$ copies of d_1 contained in d_2. In the d_1 covering of T, these full copies of d_1 are utilized in the reducible leaves of d_2 and then the remaining $|d_2| \ (mod\ |d_1|)$ vertices are all utilized by another single copy of d_1. Let these remaining vertices of the d_2 reducible leaf and the connected graph formed by including the adjacencies amongst these vertices be called r_1. We then have the following image of d_2:

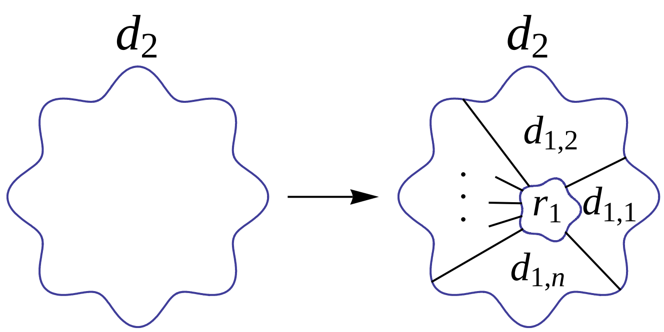

Figure 25.

Refer to this copy of d_1 that claims r_1 in a d_2 reducible leaf of T as d_{1S}. Let each of the various regions claimed by d_{1S} from other copies of d_2 in T be labeled r_i. With reference to the previous two images, we then can construct the following image of T:

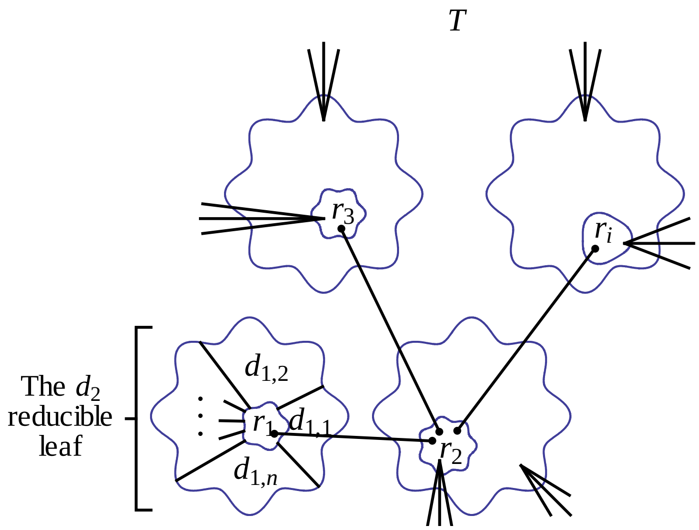

Figure 26.

We also know that d_1 must be identical to d_{1S}, which allows us to draw the following image of d_1:

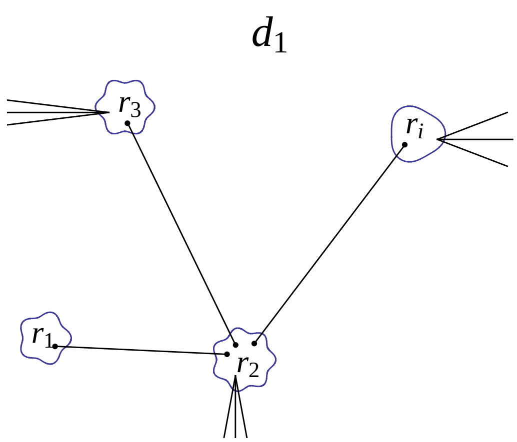

Figure 27.

We now wish to show that in d_1, all of the edges that join the various r_i together in T belong to the earlier mentioned set of edges E. This would imply that any constituents contained in all of the r_i are preserved when separating them into their own components by deleting those edges that join them together in T. This would also imply that each r_i can be "assembled" by adding edges between some number of the constituents from C. We know that all of the edges deleted to produce the d_1 covering onto T are a subset of E. If we can then show that the r_i in each d_1 have orders that are multiples of G, this would show that the desired edges that join each of the r_i in d_1 are also in E.

Lemma: Each r_i has order that is a multiple of G.

Sub-proof: We now wish to show that each of the other r_i in d_{1S} have orders that are multiples of G. Let t_i be the subgraph of T that contains r_i from d_{1S} and all other vertices that can be joined by a path in T that does not include any other vertices from d_{1S} (and obviously all of the edges that connect them). When we separate d_{1S} into its own component from the original graph T, t_i is separated into its own component(s) as well. This is because if this was not true, then there would exist a path between some vertex in t_i and some other vertex in another similarly defined subgraph of T, t_j. This hypothetical path P would also utilize no vertices in d_{1S}, and would exist in T without any of T's edges being deleted. In T however, there also exists a path between these earlier two mentioned vertices in t_i and t_j that utilizes vertices from r_i and r_j since r_i is in t_i and r_j is in t_j and since any two vertices in d_{1S} can be joined by a path. This implies the existence of a cycle in T, which is not possible.

We know t_i consists of the vertices of r_i, the other vertices of the copy of d_2 that it is contained in, and some number of full copies of d_2 as pictured below,

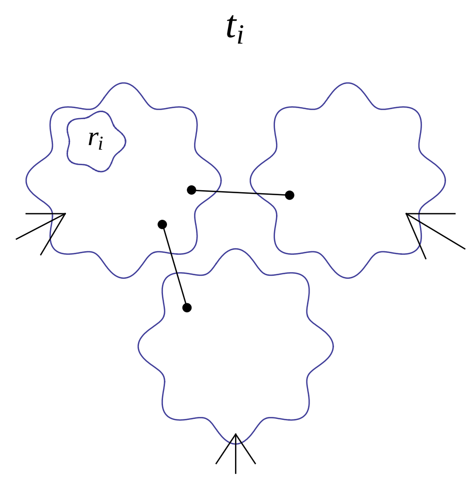

Figure 28.

This implies that t_i has order that is a multiple of G. If we remove d_{1S} from T into its own component by edge deletion, the remaining vertices of t_i not in r_i must have order that is a multiple of G since d_1 is a divisor of T. If they did not, then the divisor covering of d_1 onto T is not possible; that is d_1 cannot cover the remaining component. This implies that r_i must have order that is a multiple of G. ■

With this, we may now conclude that each r_i in T can be constructed by adding edges between constituents from C. Let C_i denote the set of constituents that can have edges added between them to produce r_i from the set C. C_i is unique since the edges of E are unique. Suppose that for some i, C_i contains an element that is not isomorphic to any of the elements in C_1.

Regardless of what order we delete the edges in E from T, we should produce a collection of constituents, each of which can map 'one to one' to an isomorphic element of C (in other words, we should produce C regardless of the order that we remove the edges of E from T). Also, we know that since d_1 is a vertex divisor of T, C should contain $|T|/|d_1|$ copies of each element that belongs to each of the C_i (this is equivalent to deleting E by first breaking into the d_1 divisor covering, separating each r_i into its own component, and then dividing edges to produce each constituent of each c_i).

Now, delete the edges of E in the following manner: first delete all edges necessary to produce the d_2 covering of T. This gives the following graph:

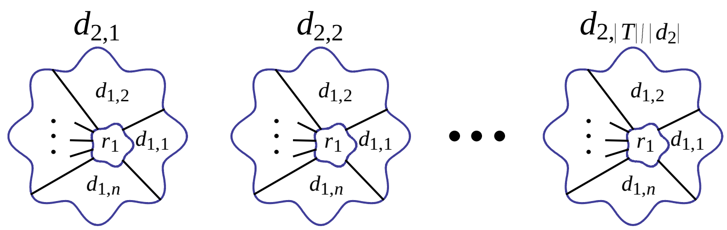

Figure 29.

We can then delete all of the $\lfloor |d_2|/|d_1| \rfloor$ copies of d_1 from each of the d_2 into their various own copies.

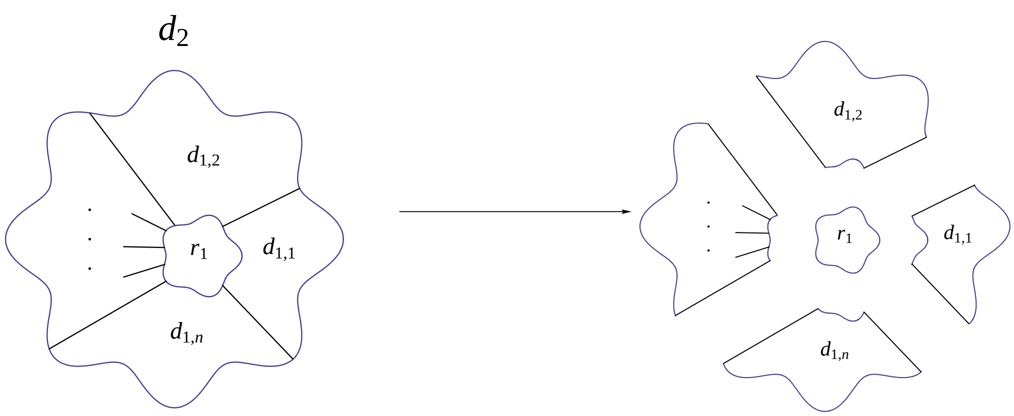

Figure 30.

The edges deleted in doing so must be in E as the produced components have order divisible by G. These produced copies can then have edges deleted to reduce each of d_1 into their various constituents, each of which clearly map to an element in C. After doing so, the constituents that remain to be mapped are those that compose $|T|/|d_2|$ copies of r_1. The remaining elements that need to be mapped in C should all be of the constituents that correspond to integer multiples of d_1, since at the moment, we have only mapped some number of the constituents from less than $|T|/|d_1|$ copies of d_1 to C.

This implies that the remaining elements in C include some number of the elements in C_i not isomorphic to any elements in C_1. The only remaining constituents to be mapped however are those that comprise r_1 and are resultantly elements of C_1, implying the mapping cannot be produced. This is a contradiction since we have only deleted the unique edges of E in this process. It follows that all C_i must contain elements isomorphic to some constituent in r_1. It can also be concluded that the remaining $|T|/|d_1|$ copies of r_1 can be divided into their constituents and those constituents can be used to assemble some number of full copies of d_1.

We do not know how many of each type of non-isomorphic constituent are in r_1. Break into two cases: There is only one constituent in r_1 to which all other constituents in r_1 are isomorphic to, or, there is at least one constituents of r_1 that is not isomorphic to another constituent of r_1. In the first case, it is easy to show that said constituent is a divisor of both d_1 and d_2 implying it is then a divisor or T. Since this constituent has order that is a multiple of G and since the order of this constituent must divide $|d_1|$ and $|d_2|$, it must in fact have order G (since you cannot have a common divisor greater than the greatest common divisor).

Suppose the second case is true. We know that r_i must contain at least two non-isomorphic constituents. Let k_1 be the first constituent, let k_2 be another constituent that is not isomorphic to k_1, let k_3 another constituent not that is not isomorphic to either k_1 or k_2…, and finally let k_j be the final constituent not isomorphic to any of the previously mentioned constituents that compose r_i.

From earlier in this proof (the part where we showed that each c_i must have constituents isomorphic to some constituent in c_i), we know that $|T|/|d_2|$ copies of r_1 can be broken into their constituents and then have edges added between them to assemble some integer number of d_1. This implies that if r_1 contains n_1 copies of k_1, n_2 copies of k_2 … and n_j copies of k_j, then d_1 must contain xn_1 copies of k_1, xn_2 copies of k_2 … and xn_j copies of k_j. In other words, the ratio of different constituents in d_1 must be proportional to the ratio of those different constituents in r_1. If this were not true, then disassembling the $|T|/|d_2|$ copies of r_1 into their constituents as wells disassembling the integer number of d_1 that those r_1's constituents can be used to assemble would yield two different collections of constituents. Because d_2 is constructed of joined copies of d_1 and a single copy of r_1, it must be the case that d_2 must also contain a number of each constituent that preserves these ratios.

This implies that for some $y = 1/GCD(n_1, n_2, \dots, n_j)$ that divides the greatest common divisor of all of the n_i, the number $y(n_1|k_1| + n_2|k_2| + \cdots + n_j|k_j|)$ must be a divisor of both $|d_1|$ and $|d_2|$. But this is a contradiction since each k_i has order that is no less than G. So it must be the case that there is only one type constituent to which all other constituents that comprise r_i must be isomorphic to. ■

There are at least two important consequences of this theorem. First, it completes a powerful classification for where vertex perfect trees can occur. From the binary divisor sieve, we know that vertex perfect trees can only occur on orders that are semi-perfect. Also, from theorem 1, we know that any vertex perfect tree has to have K_1 as a vertex divisor, which means any semi-perfect number that lacks a divisor combination that sums to itself and contains 1 does not have any vertex perfect trees. Finally, because of the interchangeable parts theorem, if these divisor combinations have two divisors but not their greatest common divisor, then that order cannot contain any vertex perfect trees.

The second important consequence of the above theorem is its implication that any two divisors of the same tree have some kind of "interchangeable part." Because the existence of two divisors d_1 and d_2 in T implies that T has a divisor d_G with order $GCD(|d_1|, |d_2|)$, and because $|d_G|$ is a divisor of $|d_1|$ and $|d_2|$, there must exist a divisor of T that is also a vertex divisor of both d_1 and d_2, that divisor being d_G. In a loose sense, this divisor d_G serves as an interchangeable part between the divisor coverings of d_1 and d_2 onto T, with either numbered divisor claiming some integer number of full copies of d_G in T. It turns out that if $|d_1|$ is a divisor of $|d_2|$, $|d_2|$ is a divisor of $|d_1|$, or if $|d_1|$ and $|d_2|$ are relatively prime, this interchangeable part is not very interesting.

As an example, examine the following tree T on 30 vertices.

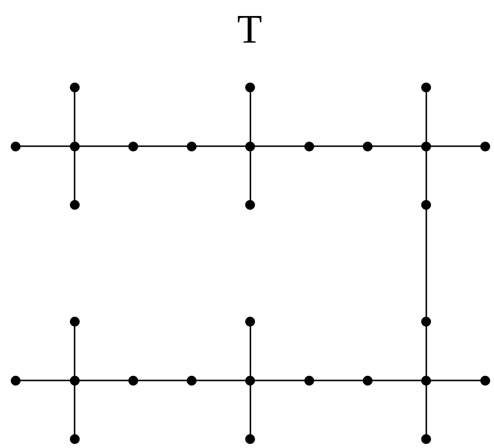

Figure 31 (a).

T has vertex divisors of order 10 and 15, and the respective divisor coverings are demonstrated as follows:

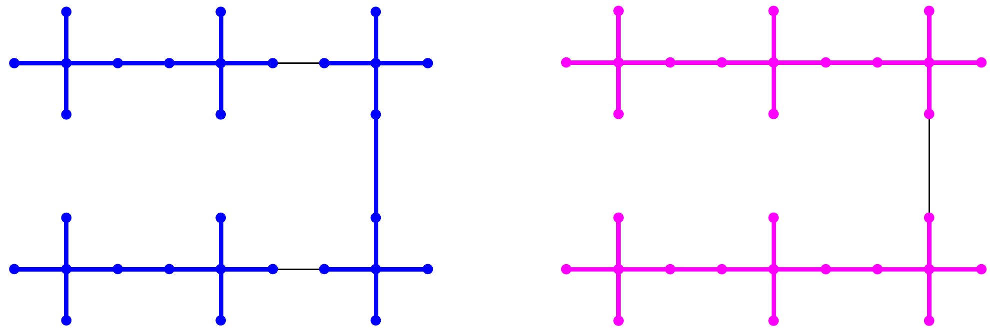

Figure 31 (b).

As we expect from the interchangeable parts theorem, *T* also has a vertex divisor of order 5 (see figure below for its vertex covering). If we look at the divisor covering of order 10 onto *T*, we can see each of the copies of the order 10 divisor claim 2 copies of the order 5 divisor as it occurs in its covering onto *T*. Similarly, each copy of the order 15 divisor claims three whole copies of the order 5 divisor.

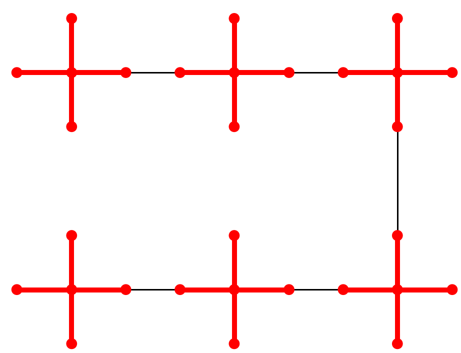

Figure 31 (c).

3.5 Vertex Perfect Tree Catalogue

We shall now produce a list of orders from 1 to 100 and whether or not those orders contain vertex perfect trees. From the binary divisor sieve, we know that the only orders that can contain vertex perfect trees are semi-perfect. So the only possible orders that contain vertex perfect trees between 1 and 100 are 6, 12, 18, 20, 24, 28, 30, 36, 40, 42, 48, 54, 56, 60, 66, 72, 78, 80, 84, 88, 90, 96, and 100. The binary divisor sieve excludes 66 and 78 since their combinations of divisors that add to 66 and 78 respectively do not include 1. A combination of the binary divisor sieve and the interchangeable parts theorem excludes the orders from having vertex perfect trees: 18, 20, 36, 42, 54, 88, and 100.

The following orders have had vertex perfect trees found on them: 6, 12, 24, 28, 30, 40, 48, 56, 60, 72, 80, 84, 90 and 96.

For a summary of the results of this section, see the table on page 77.

Chapter 4
Amicable Graphs

In this chapter we extend the generalization of this paper beyond mapping perfect numbers to graphs. As the title suggests, we will define a graph theoretic analog of amicable pairs. We will then begin an investigation of this analog in the sections that follow, focusing mainly on translating the results of chapters one and two over to the newly defined object.

4.1 Amicable graphs

In Number Theory, **amicable pairs** are pairs of positive integers with the property that each integers' proper divisors sum to the other. 220 and 284 are an example of an amicable pair as the proper divisors of 220 sum to 284 and the proper divisors of 284 sum to 220.

We define **amicable graphs** to generalize these pairs as follows: two connected graphs A_1 and A_2 are amicable if each graph's set of vertex divisors covers the other graph. A_1 and A_2 are called an **amicable pairing**.

As an example, view the following two graphs A_1 and A_2.

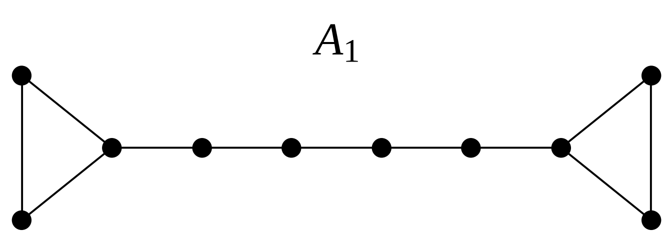

Figure 32 (a).

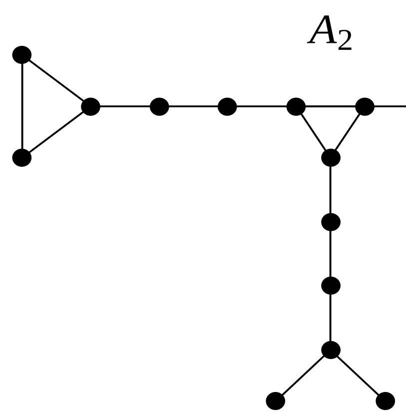

Figure 32 (b).

It can be shown that in addition to P_2 and K_1, the following connected graphs constitute the exhaustive list of vertex divisors of A_1:

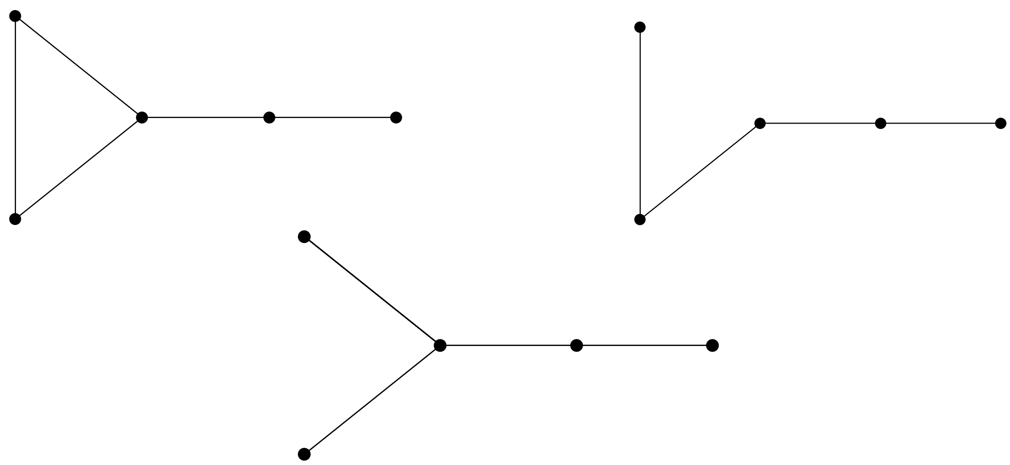

Figure 33.

We can then take the set of A_1's vertex divisors and construct A_2 by adding the following red edges.

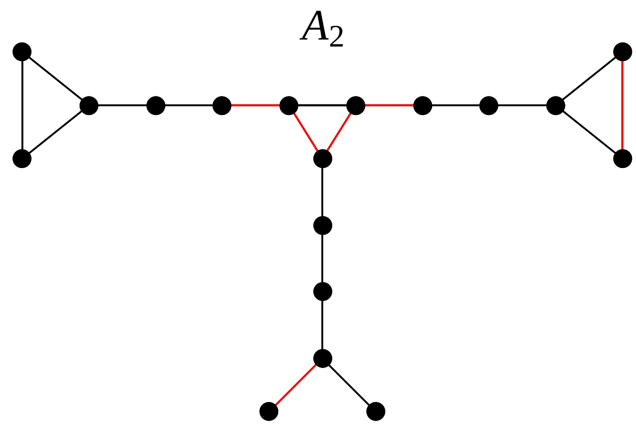

Figure 34.

It can also be shown that K_1, P_3, and the following connected graph form the exhaustive list of vertex divisors of A_2.

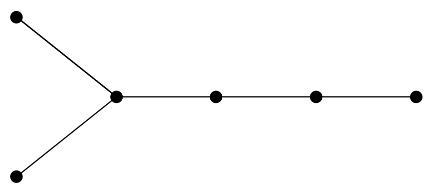

Figure 35.

Because we can then take this set of vertex divisors and construct A_1 as seen in the following image, A_1 and A_2 form an amicable pair.

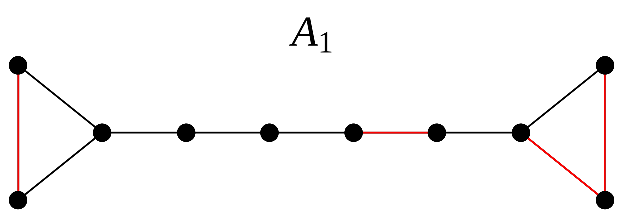

Figure 36.

The reader may have noticed that amicable pairings are not necessarily unique; that is, if A_1 and A_2 are an amicable pair, there may exist another graph A_{other} such that either A_1 and A_{other} or A_2 and A_{other} also form an amicable pair. For example, the following graph can be shown to form an amicable pair with the previous graph A_2.

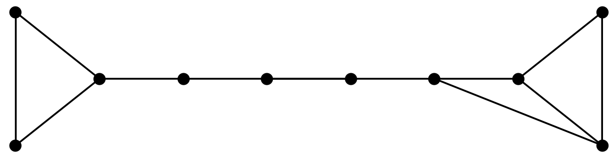

Figure 37.

This loss of uniqueness in the generalization makes amicable graphs less wieldy to work with then their number theory counterparts, which are clearly unique. It would be nice to have some property or object that characterizes a given set of similar amicable pairs. This subject is discussed more in the future work chapter of this paper (see page 72).

4.2 Early Results on Amicable Graphs

In this section, we reformat previous results on vertex perfect graphs for amicable graphs. Additionally, we also supply one new result that builds on some of these reformatted theorems. These results are simply listed without the usual narrative for the sake of brevity.

Theorem 32: If A_1 and A_2 are amicable graphs, then the sum of the orders of the divisors of A_1 must be equal to $|A_2|$ and the sum of the orders of the divisors of A_2 must be equal to $|A_1|$.

Proof: Let A_1 and A_2 be amicable graphs. By definition, A_1's vertex divisors must cover A_2 and A_2's vertex divisors must cover A_1. This implies that A_1 and A_2 can be produced by adding edges between the set of each other's divisors. Because adding edges does not increase the order a graph, the individual orders of A_1's vertex divisors must add to $|A_2|$ and the individual orders of A_2's divisors must add to $|A_1|$. ■

Amicable Divisor Sieve 33: Let A_1 and A_2 be connected graphs and let $\{q_{1,1}, q_{1,2} \dots q_{1,m}\}$ be the proper divisors of $|A_1|$ and $\{q_{2,1}, q_{2,2} \dots q_{2,m}\}$ be the proper divisors of $|A_2|$. Also, let $n_{1,i}$ be the number of vertex divisors A_1 has of order $q_{1,i}$ and let $n_{2,i}$ similarly defined for A_2. If A_1 and A_2 are amicable, then the following equations must hold:

$$1 + n_{1,1}q_{1,1} + n_{1,2}q_{1,2} + \cdots + n_{1,m}q_{1,m} = |A_2|$$

$$1 + n_{2,1}q_{2,1} + n_{2,2}q_{2,2} + \cdots + n_{2,m}q_{2,m} = |A_1|$$

Proof: By theorem 32, we know that the orders of A_1 and A_2's vertex divisors must sum to the order of the other graph, implying the existence of the above two equations. By theorems 1 and 2, each graph will have K_1 as a vertex divisor and the other divisors of each graph must have orders that properly divide the order or their original graph, implying the left hand sides of both of the above equations. ■

Theorem 34: K_1 is not an amicable graph.

Proof: As K_1 has no vertex divisors, the set of its vertex divisors cannot be used to cover another graph. ■

Theorem 35: If $|G|$ is prime, then G is not an amicable graph.

Proof: Suppose G is amicable and that $|G|$ is prime. Then there must exist some other graph that can be covered by the set of divisors of G and also whose vertex divisors cover G. By the Amicable Divisor Sieve, the sum of the orders of the vertex divisors of G must be equal to the order of this other graph. Because $|G|$ is prime, the only vertex divisor that G can have is K_1, a graph that can only cover itself. This implies that if G is amicable, it must be paired with K_1. By theorem 34, that is not possible. ■

Theorem 36: If n and m are amicable numbers, then P_n, C_n, and W_n are all amicable graphs with P_m, C_m, and W_m.

Proof: Let n and m be an amicable pair. By the path lemma and the techniques used in proving theorems 14 and 15, we know that the exhaustive list of vertex divisors of any path, wheel, and cycle includes all paths with orders that properly divide the path, wheel or cycle's order. It follows that the sum of the orders of the vertex divisors of P_n, W_n, and C_n will equal m and that the sum of the orders of the vertex divisors of P_m, W_m, and C_m will equal n. Take the set of paths whose order properly divide n and join them so as to produce P_m (clearly this can be done). Then label P_m as seen in the following image:

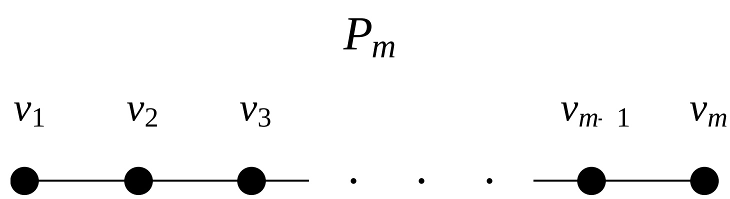

Figure 38.

If we add an edge between v_1 and v_m, then we produce C_m. If we add an edge between v_1 and v_{m-1} and then make v_m adjacent to every other vertex in the graph, we then have produced W_m. This implies that the divisors of P_n, W_n, and C_n, cover P_m, W_m, and C_m. A similar argument shows that the latter three graphs vertex divisors cover the former three graphs. Hence, P_n, C_n, and W_n are all amicable graphs with P_m, C_m, and W_m. ■

Theorem 39: The smallest amicable graph that can be paired with another graph of differing order has order 10. Furthermore, the smallest order graph that an order 10 amicable graph can be paired with is of order 18.

Proof: In the previous section, the two example graphs A_1 and A_2 (see figure 32a and 32b) were shown to be amicable graphs with respective order 10 and 18. To complete this proof, we will now show that there are no amicable graphs of order less than 10 and that there are no graphs that can be of order less than 18 that 10 can be paired with.

By theorem 34, there are no amicable graphs of order 1. By theorem 35, there are no amicable graphs of order 2, 3, 5, or 7. By the Amicable Divisor Sieve, the only possible graphs that a graph of order 4 could be amicably paired with would be an amicable graph of orders 1 or 3, neither of which will work by theorems A3 and A4. Besides a trivial pairing, the amicable divisors sieve says that the only orders that a graph of order 6 could be amicably paired with include orders 3, 4, 7, and 9. We have already shown that orders 3, 4 and 7 are not amicable. By the amicable divisor sieve, there does not exist an order 9 graph that can be paired with an order 6 graph since there are no non-negative integer solutions for c in the following equation:

$$1 + 3c = 6$$

Keeping in mind that there are only six different connected graphs of order 4 of which only one does not contain P_2 as a subgraph, by the amicable divisor sieve, the only possible orders that an amicable graph of order 8 could be paired with include 1, 3, 5, 7, 11, 15, 19, 23, 27. By theorems 34 and 35, we can rule out the orders 1, 3, 5, 7, 11, 19, and 23 from having an amicable pairing with a graph of order 8. By the amicable divisor sieve, there are no amicable pairings between a graph of order 15 and order 8 since there are no non negative integer solutions for c_1 and c_2 in the following equation:

$$1 + 3c_1 + 5c_2 = 8.$$

Similarly, by the amicable divisor sieve, we can conclude that there are no amicable pairings between any graphs of order 8 and 27 since there are no non-negative integer solutions for c_1 and c_2 in the following equation:

$$1 + 3c_1 + 9c_2 = 8.$$

By the amicable divisor sieve, if there is an amicable graph of order 9, it can only be paired with any graph of the following orders: 1, 4, and 7. From earlier results in this proof, we know that none of these orders are able to be paired with such a graph.

The only orders less than 18 that a graph of order 10 could be amicably be paired with, by the amicable divisor sieve, include 1, 3, 6, 8, 11, 13, and 16. By theorems 34 and 35, we can rule out the orders 1, 3, 11, and 13. Because there are no non-negative integer solutions to the constants in the following equations, the amicable divisor sieve rules out pairings between a graph of order 10 with a graph of order 6, 8, or 16:

$$1 + 2c_1 + 3c_2 = 10$$

$$1 + 2c_1 + 4c_2 = 10$$

$$1 + 2c_1 + 4c_2 + 8c_3 = 10$$

This completes the proof. ■

Chapter 5
Future Work

5.1 Possible Topics of Interest

To conclude this paper, we list some topics that may lend themselves well to new research projects. The first topic we list concerns an ordering of the total number of vertex divisors per order of a graph and is summarized in the following question: "If a graph has nonzero c_1 total vertex divisors of order n_1 and nonzero c_2 total vertex divisors order n_2 and $n_1 < n_2$, then is $c_1 \leq c_2$?" It seems like the answer should be no, but we have yet to find a counterexample to this trend, and the binary divisor sieve provides a "yes" to this question in the limited case of trees. If the answer to this question is in fact yes, then finding a proof would probably make for an interesting research project. In fact, if one could find a proof, such a result would imply an incredible amount of structure on connected graphs and would hopefully generate a lot of interest in this generalization. If the answer to the above question is no, then finding a counterexample to the trend would also make for an interesting research project, especially a vertex perfect counter example. Furthermore, it would be great to classify when the answer to this question is yes and when it is no for broad types of graphs.

Another topic of interest concerns perfect generation. It appears that most vertex perfect graphs are able to generate new vertex perfect graphs through the technique of perfect generation. The question we pose is this: "using the idea of perfect generation, can every vertex perfect graph produce a new vertex perfect graph?" If so, finding a proof would make for an interesting research topic and it would also be nice if one could classify "organically" appearing vertex perfect graphs (VPGs that are not produced through generation) vs "artificially" generated vertex perfect graphs. If vertex perfect graphs exist that makes the answer to the posed question "no", it would be nice to find at least one and classify under what circumstances such vertex perfect graphs occur. Conversely, finding circumstances under which perfect generation is possible would also be interesting. It may be a good idea to investigate this topic in the limited case of trees for extra "proof leverage" before confronting all connected graphs.

A final topic concerning vertex perfect graphs is related to the divisor sieve. Every graph that has been found in this project that satisfies the divisor sieve also has had a perfect covering. It may be that the converse of the divisor sieve is true: if a graph satisfies the divisor sieve, then it has a perfect covering (implying that it is then vertex perfect). Either proving this conjecture or searching for some contrived counterexample would probably both make engaging research projects. This topic may also lend itself well to first being examined in the limited case of trees before being extended to connected graphs as a whole.

Amicable graphs may also show promise as a research topic. At the moment, little is known about them besides the few results that have been extended to them in this paper. Furthermore, much work is needed in defining and classifying these new objects to account for trivial properties (i.e., by our definition of an amicable graph, all vertex perfect graphs are technically amicable with themselves). It would also be interesting to find a unique property or properties that unifies shared amicable pairs.

Similar to amicable graphs, another topic we recommend deals with yet another generalization of the number theory generalization to graphs. For someone looking for a completely uninvestigated idea, mapping multi-perfect numbers to connected graphs may make for an interesting research project. In number theory, a k multi-perfect number is a positive integer n with the property that its proper divisors sum to kn. The analog of a k multi-perfect number is called a k vertex perfect graph, and is defined as follows: if the vertex divisors of a graph G can cover k copies of G, then G is a k vertex perfect graph.

The final idea we suggest as a research topic concerns a different generalization of perfect numbers to graphs. While the generalization of interest in this paper largely maps perfect numbers to the vertices of graphs (hence the name vertex perfect graph), another generalization has been proposed that maps perfect numbers to the edges of graphs. The graph theoretic analog of a perfect number in this other generalization is referred to as an edge perfect graph. We define an edge perfect graph as a connected graph that can be covered by its edge divisors. An edge divisor D of a graph G is then a graph with that property that multiple copies of D can cover G (in this case, the multiple copies of D can share vertices but not edges, an important difference between the two generalizations). The following image is of an example edge perfect graph P_7 shown with it being covered by the set of its edge divisors:

Figure 39.

Appendix 1

The Vertex Perfect Graph Catalogue

Order	VPG Exist	Order	VPG Exist	Order	VPG Exist
1	No	**35**	?	**69**	No
2	No	**36**	?	**70**	?
3	No	**37**	No	**71**	No
4	No	**38**	No	**72**	Yes
5	No	**39**	No	**73**	No
6	Yes	**40**	Yes	**74**	No
7	No	**41**	No	**75**	No
8	No	**42**	Yes	**76**	?
9	No	**43**	No	**77**	?
10	No	**44**	?	**78**	Yes
11	No	**45**	Yes	**79**	No
12	Yes	**46**	No	**80**	Yes
13	No	**47**	No	**81**	No
14	No	**48**	Yes	**82**	No
15	No	**49**	No	**83**	No
16	No	**50**	No	**84**	Yes
17	No	**51**	No	**85**	?
18	Yes	**52**	No	**86**	No
19	No	**53**	No	**87**	No
20	Yes	**54**	Yes	**88**	?
21	Yes	**55**	Yes	**89**	No
22	No	**56**	Yes	**90**	Yes
23	No	**57**	No	**91**	?
24	Yes	**58**	No	**92**	?
25	No	**59**	No	**93**	No
26	No	**60**	Yes	**94**	No
27	No	**61**	No	**95**	?
28	Yes	**62**	No	**96**	Yes
29	No	**63**	No	**97**	No
30	Yes	**64**	No	**98**	?
31	No	**65**	?	**99**	?
32	No	**66**	?	**100**	?
33	No	**67**	No		
34	No	**68**	No		

Appendix 2

The Vertex Perfect Tree Catalogue

Order	VPT Exists	Order	VPT Exists	Order	VPT Exists
1	No	35	No	69	No
2	No	36	No	70	No
3	No	37	No	71	No
4	No	38	No	72	Yes
5	No	39	No	73	No
6	Yes	40	Yes	74	No
7	No	41	No	75	No
8	No	42	No	76	No
9	No	43	No	77	No
10	No	44	No	78	No
11	No	45	No	79	No
12	Yes	46	No	80	Yes
13	No	47	No	81	No
14	No	48	Yes	82	No
15	No	49	No	83	No
16	No	50	No	84	Yes
17	No	51	No	85	No
18	No	52	No	86	No
19	No	53	No	87	No
20	No	54	No	88	No
21	No	55	No	89	No
22	No	56	Yes	90	Yes
23	No	57	No	91	No
24	Yes	58	No	92	No
25	No	59	No	93	No
26	No	60	Yes	94	No
27	No	61	No	95	No
28	Yes	62	No	96	Yes
29	No	63	No	97	No
30	Yes	64	No	98	No
31	No	65	No	99	No
32	No	66	No	100	No
33	No	67	No		
34	No	68	No		

Appendix 3

<u>Examples of Vertex Perfect Graphs for Established Orders</u>

$|G| = 6$
See figure 6.

$|G| = 12$
See figure 7.

$|G| = 18$
See figure 12.

$|G| = 20$

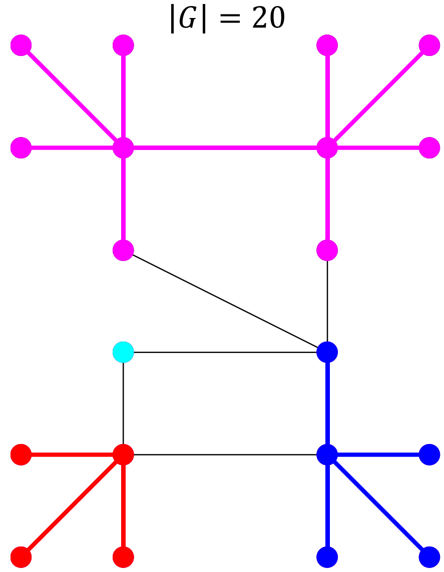

Figure 40.

$|G| = 21$

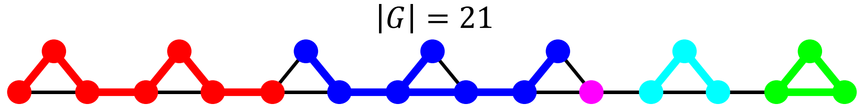

Figure 41.

$|G| = 24$

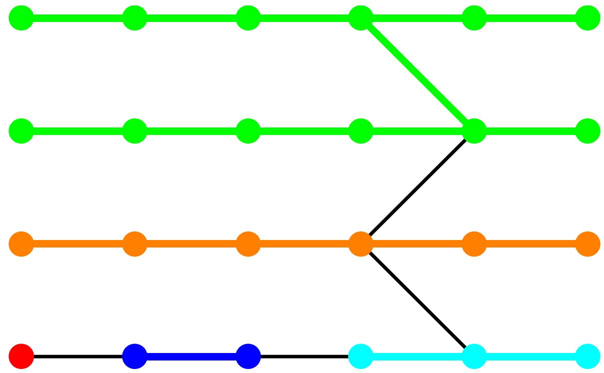

Figure 42.

$|G| = 28$

See figure 6.

$|G| = 30$

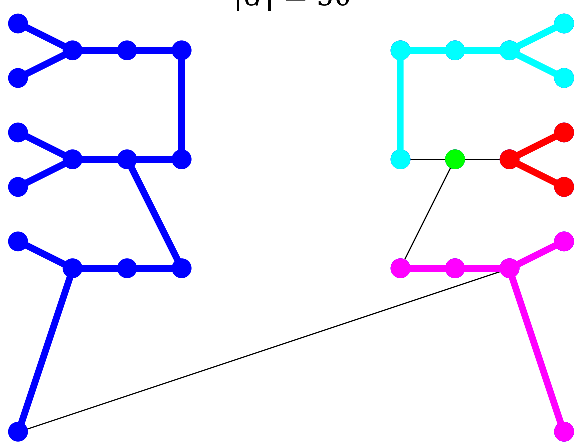

Figure 43.

$|G| = 40$

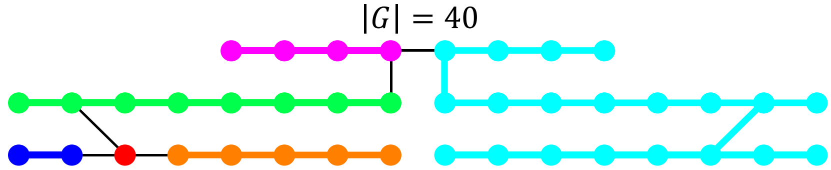

Figure 44.

$|G| = 42$

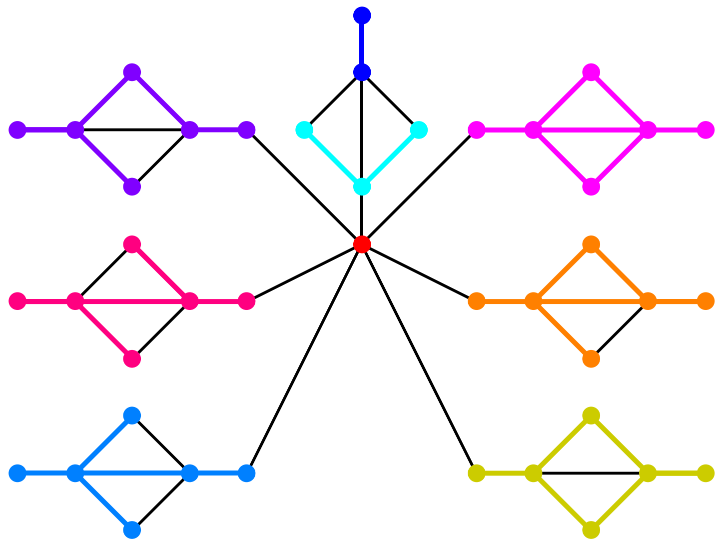

Figure 45.

$|G| = 45$

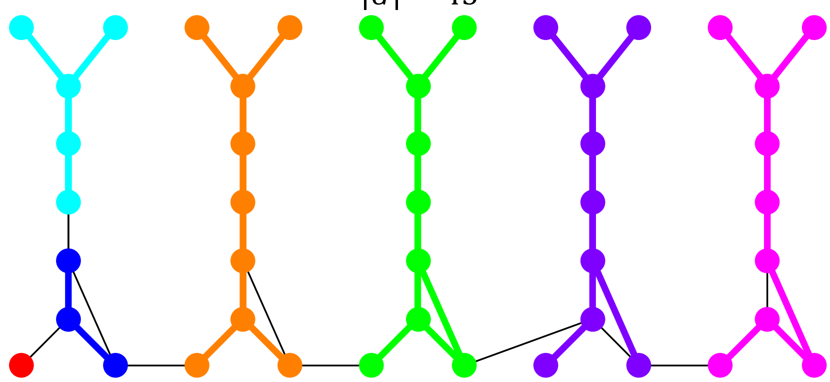

Figure 46.

$|G| = 48$

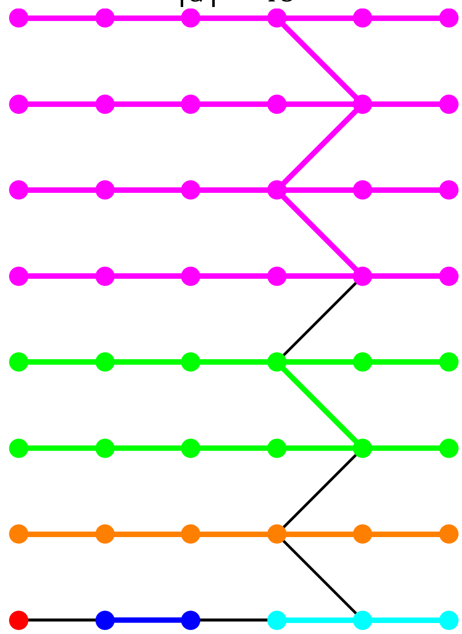

Figure 47.

$|G| = 54$

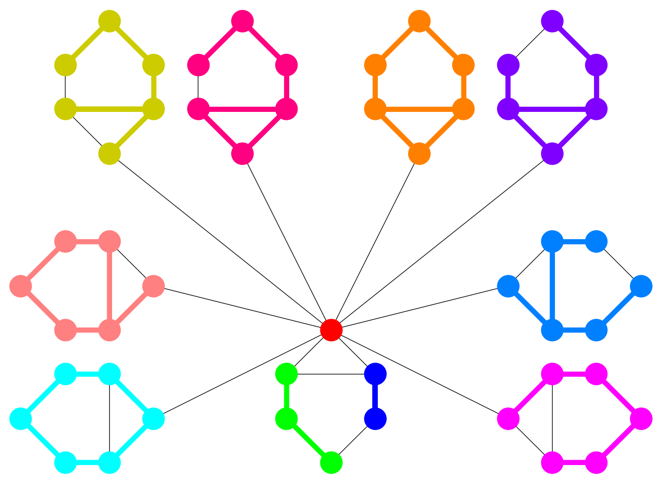

Figure 48.

$|G| = 55$

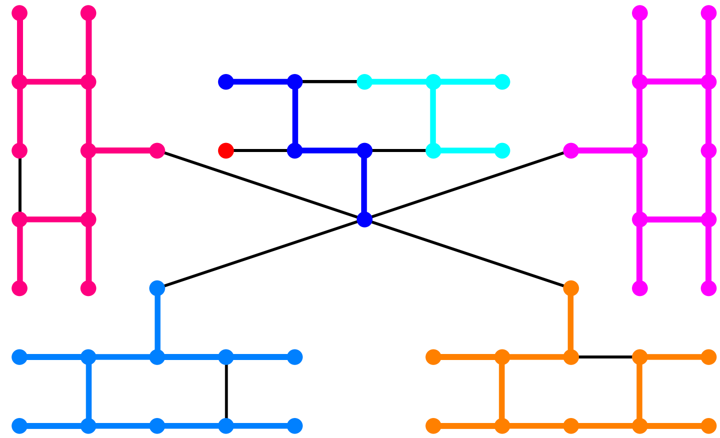

Figure 49.

$|G| = 56$

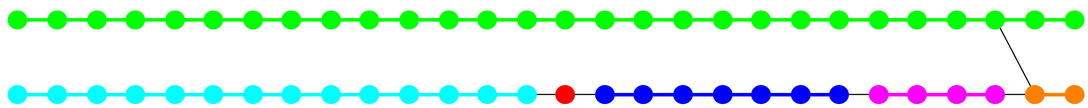

Figure 50.

$|G| = 60$

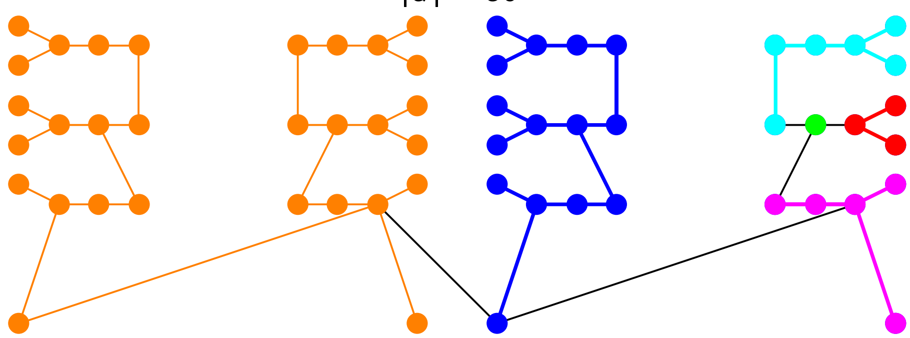

Figure 51.

$|G| = 72$

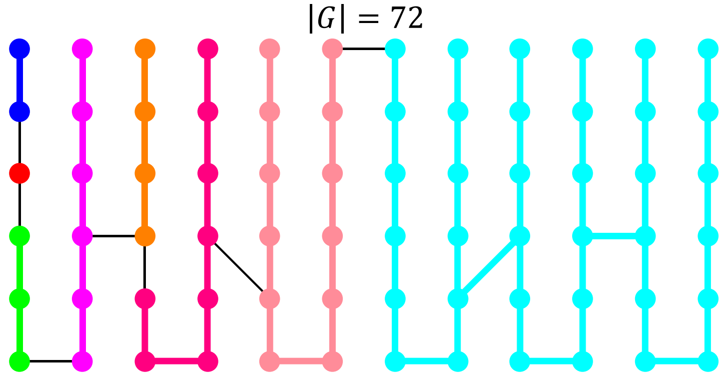

Figure 52.

$|G| = 78$

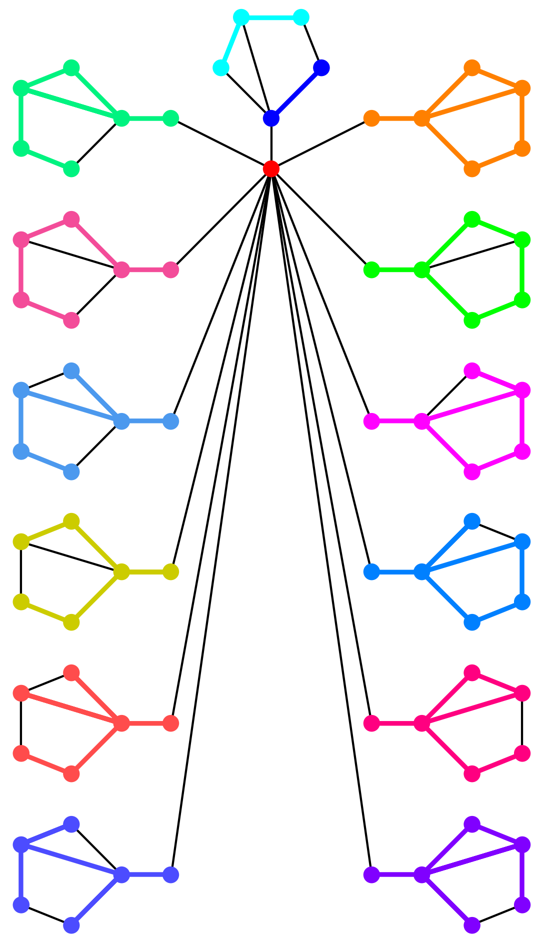

Figure 53.

$|G| = 80$

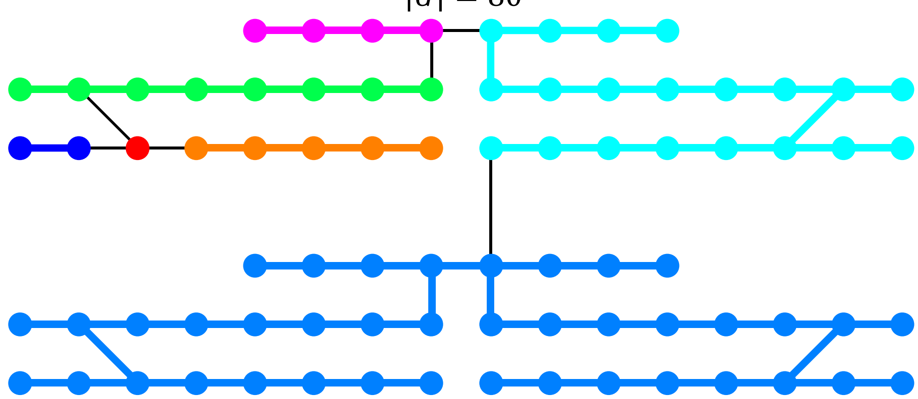

Figure 54.

$|G| = 84$

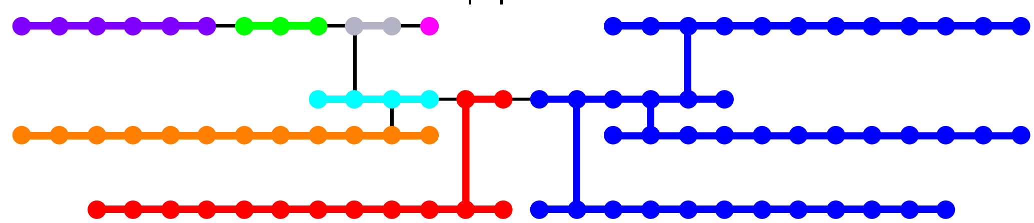

Figure 55.

$|G| = 90$

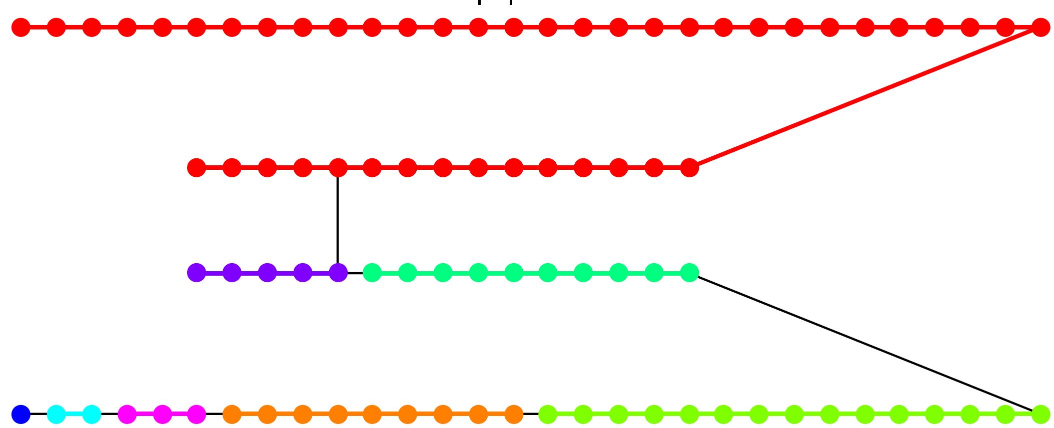

Figure 56.

$|G| = 96$

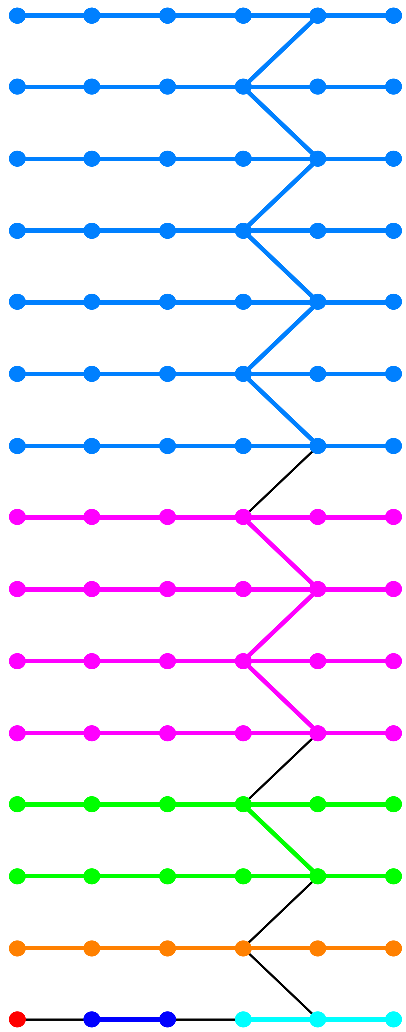

Figure 57.

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[B2] Weisstein, Eric W. "Perfect Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/PerfectNumber.html

[B3] Private Correspondence between Erich Friedman and George Sicherman, August 2014.