A STUDY OF SEMIREGULAR GRAPHS

By
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ABSTRACT

A STUDY OF SEMIREGULAR GRAPHS

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Just as a regular graph is one in which each vertex is distance 1 away from exactly the same number of vertices, we define a semiregular graph to be a graph in which each vertex is distance 2 away from exactly the same number of vertices. If each vertex of a semiregular graph is distance 2 away from n vertices, we say that that graph is n-semiregular. We give examples of semiregular graphs, describe the barbell class, and describe how the property of semiregularity relates to other properties of graphs, such as regularity, vertex-transitivity and symmetry. The classes of 0-semiregular graphs, 1-semiregular graphs, 2-semiregular graphs and the semiregular trees are fully classified. In addition, an algorithm for determining whether a graph is semiregular is presented.

CHAPTER 1
GRAPH THEORY CONCEPTS

1.1. GRAPHS

In graph theory, a **graph** is defined to be a set of points called **vertices** that are connected by lines called **edges**. Figure 1 shows some examples of graphs under this definition.

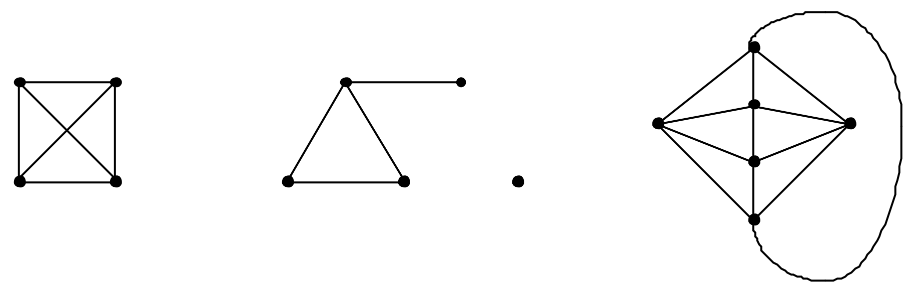


Figure 1

We will consider only simple graphs; that is, there is a maximum of one edge connecting any two vertices, and there can be no edges connecting a vertex to itself.

The **vertex set** of a graph is the set of all the vertices in the graph. Likewise, the **edge set** of a graph is the set of all edges in the graph. The vertex and edge sets of a graph G are denoted $V(G)$ and $E(G)$, respectively. The **degree** of a vertex v, denoted $\deg(v)$, is the number of edges that contain v.

A **path** between two vertices a and b is a sequence of vertices and edges that lead from a to b in which no vertex is repeated. The **length** of a path is the number of edges that it contains. Figure 2 shows a path of length 5 from a to b.

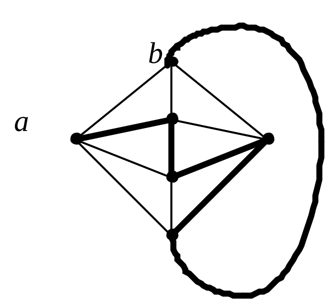


Figure 2

A graph of n vertices that is just a single path is denoted $\mathbf{P}_n$. The **distance** between two vertices a and b is the number of edges contained in the shortest path between a and b. Referring back to Figure 2, a and b are distance 1 apart.

A graph H is a **subgraph** of a graph G if the vertex set of H is contained within the vertex set of G and two vertices are connected in H only if they are connected in G. Figure 3 shows a graph, G, and one possible subgraph, H.

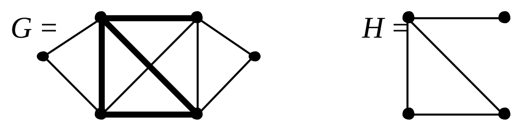


Figure 3

When two vertices are connected in H *if and only if* they are connected in G, then H is called an **induced subgraph** of G.

The **complement**, $\overline{G}$, of a graph G has the same vertex set as G and vertices v_1 and v_2 are connected in $\overline{G}$ if and only if they are not connected in G. Figure 4 shows a graph and its complement.

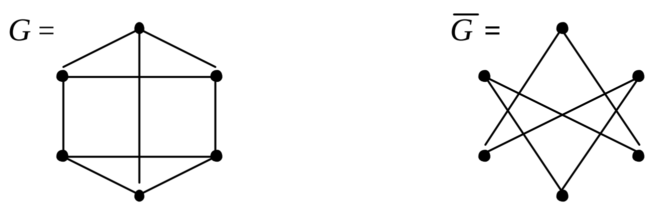


Figure 4

1.2. PROPERTIES OF GRAPHS

A graph is **connected** if there exists a path connecting any two vertices of a graph. Otherwise, the graph is **disconnected**. Each connected part of a disconnected graph is called a **component**. Figure 5 shows both a connected and a disconnected graph.

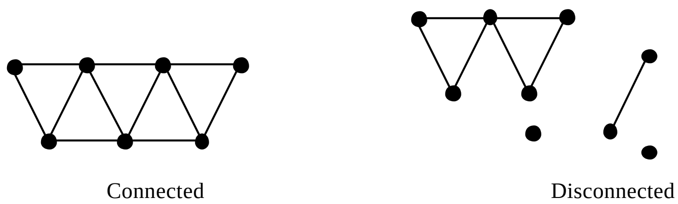


Figure 5

A graph is **regular** if every vertex in the graph has the same degree. If all the vertices of a graph have degree n, we call that graph ***n*-regular**. Figure 6 shows some examples of regular graphs:

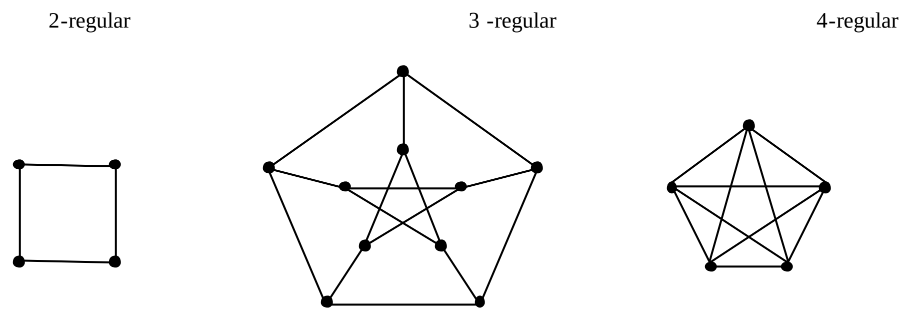


Figure 6

A **cycle** is a path of length 2 or more with its endpoints joined by an additional edge. A cycle with n vertices is called an ***n*-cycle**. The cycles are exactly the 2-regular graphs. Figure 7 shows some examples of cycles.

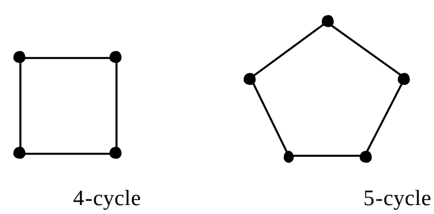


Figure 7

A graph is called **tree** if it is connected and contains no cycles as subgraphs. Figure 8 shows two examples of trees.

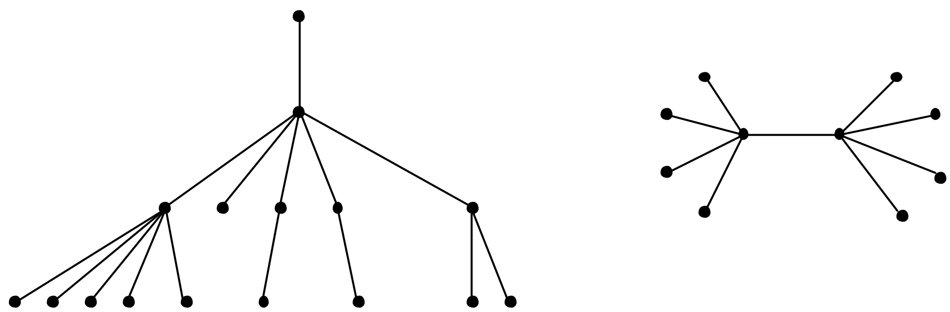


Figure 8

A graph is called **complete** if every vertex is connected to every other vertex in the graph. $\mathbf{K}_{\boldsymbol{n}}$ denotes the complete graph with *n* vertices. Figure 9 shows some examples of complete graphs.

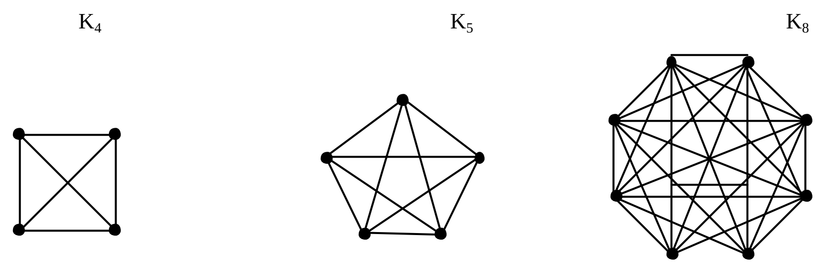


Figure 9

CHAPTER 2
SEMIREGULAR GRAPHS

2.1. INTRODUCTION

A graph is **semiregular** if each vertex in the graph is distance 2 away from exactly the same number of vertices. If each vertex is distance 2 from n other vertices, we call that graph ***n*-semiregular**. Figure 10 shows some examples of semiregular graphs.

2-semiregular 3-semiregular 6-semiregular

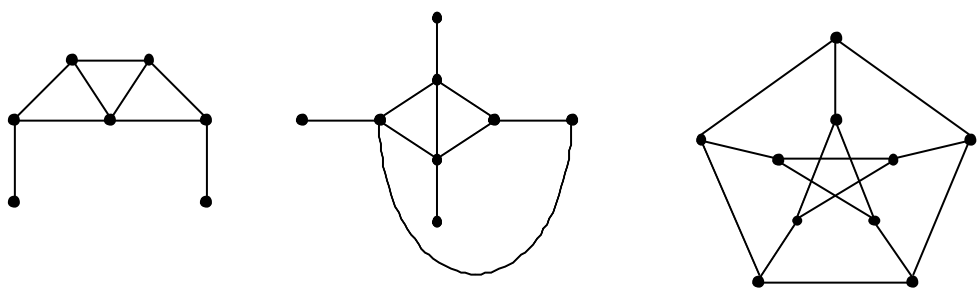


Figure 10

Semiregular graphs are a natural extension of the idea of regular graphs. Although extensive literature exists on regular graphs, semiregular graphs have been much less studies. A similar idea appears in "Distance Degree Regular Graphs" by Bloom, Kennedy and Quintas.

Define $\mathbf{deg_2(\mathit{v})}$ to be the number of vertices that are distance 2 away from v in a given graph. It is obvious that the union more than one n-semiregular graph is also n-semiregular, so we will limit our discussion to connected semiregular graphs.

2.2. THE BARBELL CLASS

The ***n*-barbell graph** is formed by taking a connected pair of vertices, v_1 and v_2 and then connecting n new vertices to v_1 and then n new vertices to v_2. Figure 11 depicts several of the smaller barbell graphs.

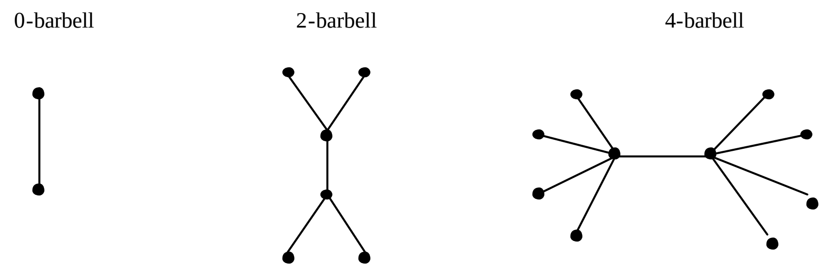


Figure 11

Theorem 2.1. The n-barbell graph is n-semiregular for all $n \geq 0$.

Proof. Let G be the n-barbell graph. That is, G is formed by a central line segment connecting v_1 and v_2 with n other vertices connected to each of v_1 and v_2. Let v be a vertex in G. For $n = 0$, G is 0-semiregular. For all other n, there are two possible cases:

Case 1. v is a point on the central line segment of G. Without loss of generality, say that v is v_1. There are n +1 vertices connected to v, including v_2. There are also n other vertices connected to v_2, and v is distance 2 from each of them. We have considered all the vertices of G, so $\deg_2(v) = n$.

Case 2. v is an endpoint of G. Without loss of generality, say that v is connected to v_1. v is distance 2 from v_2 and the other n-1 other vertices connected onto v_1. v is distance 3 from the n vertices connected onto v_2. We have considered all the vertices of G, so $\deg_2(v) = (n-1)+1 = n$.

Thus, $\deg_2(v) = n$ for every vertex v in G, and G is n-semiregular. ✪

This theorem leads us to the solution of a natural question that arises concerning semiregular graphs: Is there an n-semiregular graph for every n?

Corollary 2.2. There exists a n-semiregular graph for every $n \geq 0$.

Proof. For any $n \geq 0$, the n-barbell graph is n-semiregular. ✪

2.3. VERTEX-TRANSITIVITY

An **automorphism** of a graph G is a one-to-one, onto map f:V(G)→V(G) such that $\{u,v\} \in E(G)$ iff $\{f(u),f(v)\} \in E(G)$. A graph G is **vertex-transitive** if for all pairs of vertices v_1 and v_2 of G there is an automorphism of G mapping v_1 to v_2.

Lemma 2.3. If f is an automorphism of a connected graph G and a, b are vertices of G, then the distance between a and b is the same as the distance between $f(a)$ and $f(b)$.

Proof. Let G be a graph, and let a and b be vertices of G. Let f:V(G)→V(G) be an automorphism of G. Say that a and b are distance d apart. Then there is a path a, v_1, v_2,......, v_{d-1}, b (where $v_i \in V(G),\ \forall i$) that has length d. Now consider the vertices $f(a)$, $f(v_1)$, $f(v_2)$,....., $f(v_{d-1})$, $f(b)$. By definition of automorphism, the connections between vertices are preserved under the automorphism, so the sequence $f(a)$, $f(v_1)$, $f(v_2)$,....., $f(v_{d-1})$, $f(b)$ forms a path of length d from $f(a)$ to $f(b)$. Therefore, the distance between $f(a)$ and $f(b)$ is at most d. Say that there is a path $f(a)$, u_1, u_2,....., u_{n-1}, $f(b)$ in G such that the length of the path is n, where $n < d$. Since f is a bijection we can rewrite this path as $f(a)$, $f(w_1)$, $f(w_2)$,....., $f(w_{n-1})$, $f(b)$ for some w_1, w_2,...., w_{n-1} in G. Then a, w_1, w_2,...., w_{n-1}, b defines a path of length n in G, which contradicts our assumption that a and b are distance d apart. Thus, $f(a)$ and $f(b)$ are the same distance apart as a and b.✪

Theorem 2.4. All connected vertex-transitive graphs are semiregular.

Proof. Let G be a connected vertex-transitive graph, and let v_1, v_2 be vertices of G. Say that v_1 is distance 2 away from exactly n other vertices, namely, u_1, u_2, u_3,...., u_n. Now, since G is vertex-transitive, there exists an automorphism, φ, that maps v_2 onto v_1. Since φ is an automorphism, there must also be vertices w_1, w_2, w_3,...., $w_n \in V(G)$ such that $\varphi(w_i) = u_i$. Since, according to Lemma 2.3, distance is preserved by automorphism,

$$d(v_1, u_i) = d(v_2, w_i)$$

$$2 = d(v_2, w_i)$$

Thus, v_2 is also distance 2 away from at least n other vertices, namely, $w_1, w_2, w_3, \ldots, w_n$. Say that v_2 is distance 2 from an additional vertex, x. Since f is a bijection and because distance is preserved by automorphism, there exists a $y \in V(G)$ that is not any of the u_i and that is distance 2 from v_1. That contradicts our assumption that $u_1, u_2, u_3, \ldots, u_n$ are the only vertices that are distance 2 from v_1. Thus, v_2 is distance 2 away from exactly n other vertices. Since this is true for any $v_1, v_2 \in V(G)$, G must be n-semiregular. ✪

CHAPTER 3
CLASSIFICATIONS

3.1. THE 0-SEMIREGULAR GRAPHS

Now that we know some basic truths about semiregular graphs, we shall move on to classifying certain types of semiregular graphs. First, we shall fully classify the 0-semiregular graphs.

Theorem 3.1. A connected graph is 0-semiregular if and only if it is a complete graph, K_n, for $n \geq 1$.

Proof. Let G be a connected 0-semiregular graph with n vertices, $n \geq 1$. The distance between any two vertices of G must be 1, because a distance greater than 1 would mean that G would have two vertices that were distance 2 apart, and G would therefore not be 0-semiregular. A connected graph with n vertices in which all vertices are at distance 1 from all other vertices is the complete graph, K_n.

Let G be the complete graph, K_n, for $n \geq 1$. Then for any vertex v in G, v is not distance 2 away from any other vertices. Thus, G is 0-semiregular. ✪

3.2. THE 1-SEMIREGULAR GRAPHS

Now we shall classify the 1-semiregular graphs, but first we will need a lemma.

Lemma 3.2. Every finite 1-semiregular graph has an even number of vertices.

Proof. Let G be a 1-semiregular graph. Let v be a vertex of G. Since G is 1-semiregular, v must be distance 2 away from exactly one other vertex in G. Call it u. Likewise, u is distance 2 away from v alone. Thus the vertices of G can be divided into pairs that are distance 2 away from each other, but not from any other vertex. If G had an odd number of vertices, there would be a vertex that was not part of a pair, and therefore not distance 2 away from any other vertex. Then G would not be 1-semiregular. So G must have an even number of vertices. ✪

Now we are ready for the classification of the 1-semiregular graphs.

Theorem 3.3. A connected graph is 1-semiregular if and only if it is P_4 or $\overline{\left(\bigcup_{i=1}^{n} P_2\right)}$, for $n \geq 2$.

Proof. Let G be a connected 1-semiregular graph. G must have at least four vertices, because it must have an even number of vertices (according to Lemma 3.2), and the only connected graph with two vertices, P_2, is not 1-semiregular.

Case 1: Every two vertices of G are either distance 1 or 2 apart. Say that G has m vertices, $m \geq 4$. According to Lemma 3.2, m must be even, and its vertices can be divided into pairs that are distance 2 apart from each other. Since, as stated, the only possible distances between two vertices of G is 1 or 2, each vertex must be distance 1 away from every vertex except its pair. Thus, the graph is completely connected except that the pairs are not connected. The complement of the graph is then a union of (at least two) P_2 graphs. Thus G can be written in the form $\overline{\left(\bigcup_{i=1}^{n} P_2\right)}$ for $n \geq 2$.

Case 2: There are vertices of G that are at distance greater than 2 from one another. That means that there exists a pair of vertices, v_1 and v_4, that are distance 3 apart, as shown below:

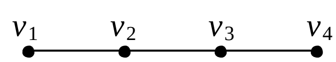


Figure 12

What we have here is actually a 1-semiregular graph (P_4). We must check to see if we can expand this graph. First we should check to see if any more edges can be added to the vertices we already have. There are only two distinct possibilities:

Figure 13

In both cases, v_1 is no longer distance 3 from v_4, which contradicts our assumption. So no more edges can be added among the vertices v_1 through v_4. Now we examine what happens when a new vertex is added to G, call it v_5. There are only two cases that must be examined.

Case 2a: We connect v_5 to an endpoint; say v_1.

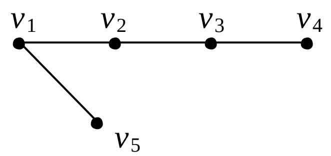


Figure 14

As can be seen in the figure above, the addition of v_5 causes v_2 to be distance 2 away from both v_4 and v_5. We cannot, by adding edges, make the distance between two vertices any longer, so we must make v_2 to be only distance 1 away from v_5.

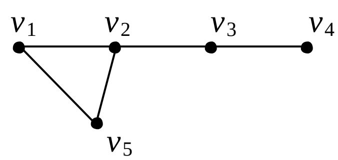


Figure 15

Now we have the problem that v_3 is distance 2 away from more than one other vertex, namely v_1 and v_5. We cannot connect v_3 to v_1, so we must connect v_3 to v_5.

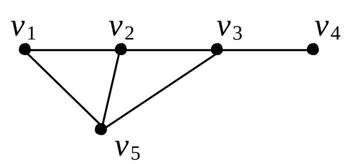


Figure 16

In the same manner, it can be concluded that v_4 and v_5 must also be connected.

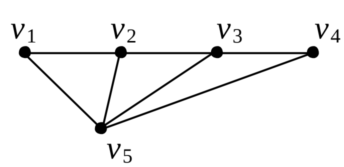


Figure 17

But now v_1 is only distance 2 away from v_4, which contradicts our assumption.

Case 2b: We connect v_5 to a non-endpoint; say v_2. Just as in Case 2a, this causes v_1 and v_3 to be distance 2 away from two vertices, so we must connect both v_1 and v_3 to v_5. Now v_4 is distance 2 away from two other vertices, so we connect v_4 to v_5. Again, there is now a path of length 2 between v_1 and v_4 so we have reached a contradiction.

Assume that G is either P_4 or $\overline{\left(\bigcup_{i=1}^{n} P_2\right)}$, for $n \geq 2$. If G is P_4, then G is connected and 1-semiregular. Say that G is a complement of a union of at least two P_2's. Then, as seen above, each vertex of G is connected to (distance 1 from) every other vertex except its 'pair'. Since $n \geq 2$, G must have at least four vertices. Let v_1 and v_2 be any of the pairs of vertices in G, and let v be a third vertex. Then v must be connected to both v_1 and v_2, creating a path of length 2 from v_1 to v_2. Since they are not connected, v_1 and v_2 are indeed distance 2 from each other. Thus G is 1-semiregular. It also follows that G is connected, since v_1 is connected to all vertices except v_2, and there is a path from v_1 to v_2. ✪

Figure 18 shows some examples of graphs of the form $\overline{\left(\bigcup_{i=1}^{m/2} P_2\right)}$.

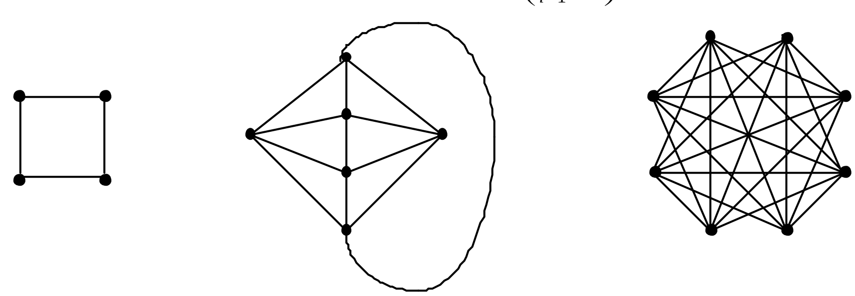


Figure 18

3.3. THE 2-SEMIREGULAR GRAPHS

Now that we have seen the classifications of the 0- and 1-semiregular graphs, we shall move on to the classification of the 2-semiregular graphs, which is a much more complicated problem.

Theorem 3.4. A connected graph is 2-semiregular if and only if it is an n-cycle or the complement of an n-cycle for $n \geq 5$, the complement of the union of at least two disjoint cycles, or one of the seventeen graphs below:

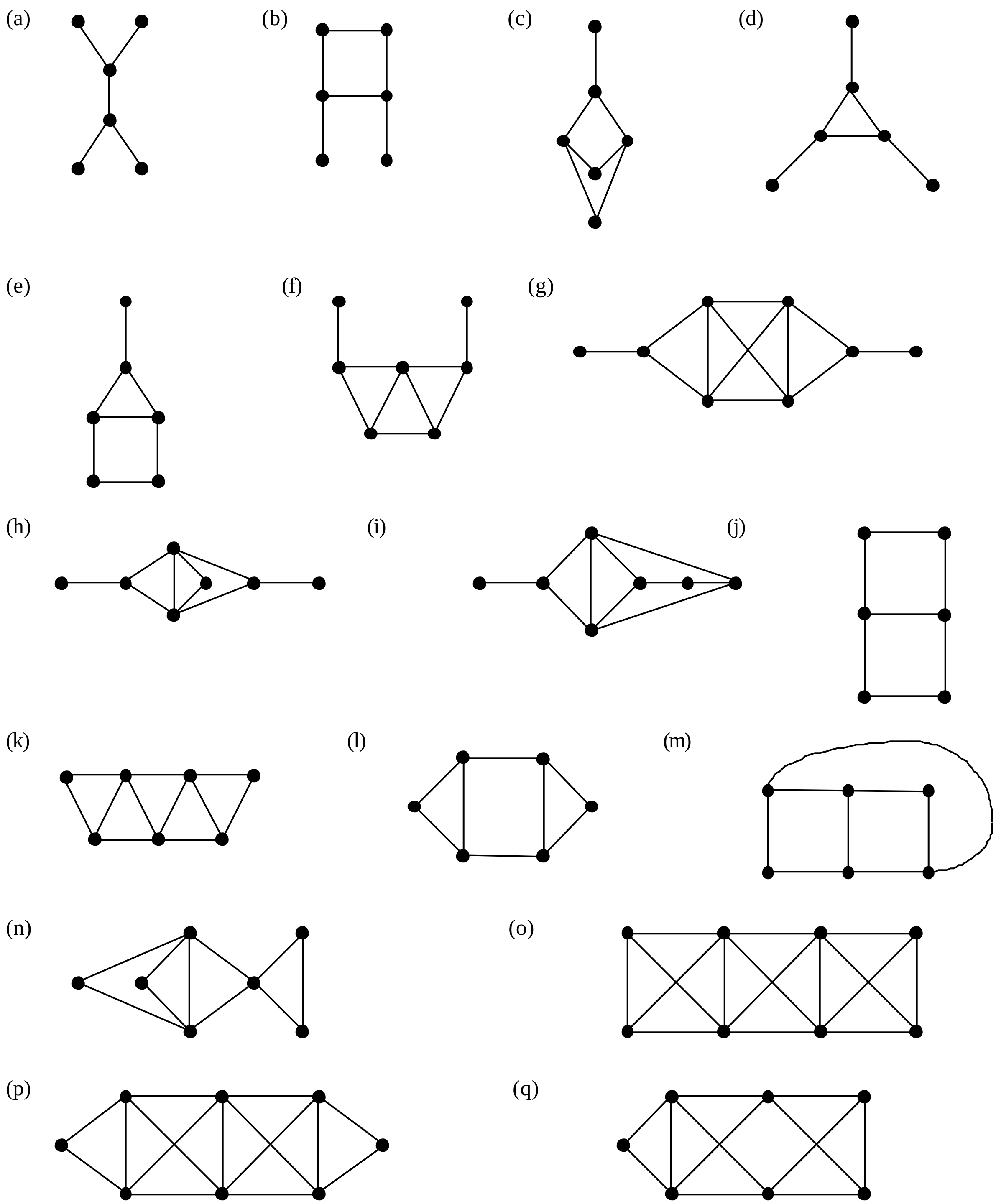


Figure 19

Proof. Let G be a connected 2-semiregular graph. We break up the possibilities for G into two cases: First, that G contains an endpoint. Second, that G does not.

If we know that G has an endpoint, we can start with that endpoint, and try all the possible cases. See Appendix A for a full exploration of all possible cases. We conclude that graphs (a), (b), (c), (d), (e), (f), (g), (h), and (i) in Figure 19 above are the only possibilities for a connected 2-semiregular graph that has an endpoint. Secondly, assume that G does not contain an endpoint. We will divide this possibility further into two cases:

Case 1: G contains no endpoint, and the distance between any two vertices of G is at most 2. That means that every vertex v in G is connected to every other vertex in G except for the two vertices that it is distance 2 away from. Now consider $\overline{G}$. $\overline{G}$ must be a 2-regular graph, but it might have more than one component. Recall that the only possible connected 2-regular graphs are the cycles. Thus $\overline{G}$ must be either a cycle or the union of two or more disjoint cycles. $\overline{G}$ cannot be a 3-cycle or a 4-cycle, because then G would be disconnected.

Case 2: G contains no endpoint, and there exist vertices u and v of G that are at least distance 3 apart. See Appendix B for a full exploration of all the possible cases. We conclude that the n-cycle graphs for $n \geq 6$ and the graphs (j), (k), (l), (m), (n), (o), (p), and (q) in Figure 19 are the only possibilities for a connected 2-semiregular graph that has no endpoint and contains two vertices that are at least distance 3 apart. (Notice that the case that G is a 5-cycle is contained in Case 1, because the complement of a 5-cycle is itself a 5-cycle.)

If G is an n-cycle for $n \geq 5$, then G is obviously connected and 2-semiregular. If G is the complement of an 5-cycle, then G is itself a 5-cycle, and is covered by the previous case. If G is the complement of an n-cycle for $n \geq 6$, then every vertex in G is connected to all but two of the other vertices. Consider a vertex v in G that is not connected to v_1 or v_2. Additionally, v_1 is not connected to u_1 and v_2 is not connected to u_2, where neither u_1 nor u_2 is v. But G has at least six vertices, so G must contain a vertex u that *is* connected to v *and* v_1 *and* v_2. Thus there is a path of length 2 between v and v_1 and between v and v_2. Thus $\deg_2(v) = 2$, meaning G is 2-semiregular. Furthermore, since v is connected to all vertices except v_1 and v_2, G is connected.

Now consider the case that G is the complement of the union of at least two cycles. Just as in the case that G was the complement of a cycle, each vertex v of G must be connected to all but two of the other vertices in G. Let u be a vertex that was part of a different cycle than v before the complement was taken. Then in G, u must be connected to both v and the two vertices that v is not connected to, creating paths of length 2 between them. Again, this causes the graph to be connected.

The seventeen sporadic examples can be checked individually for 2-semiregularity. ✪

3.4. THE SEMIREGULAR TREES

We will now classify some semiregular graphs by a different type of property.

Theorem 3.5. A finite tree is semiregular if and only if it is P_1 or a member of the barbell class.

Proof. Recall that by definition, trees are connected. Let G be a finite semiregular tree. If G has just one point (P_1), it is a 0-semiregular tree.

Figure 20

Now consider the case that G has at least two points. Since G is a finite tree, it must have an endpoint. Call it v. Vertex v must be connected to exactly one other vertex, which we will call x.

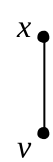


Figure 21

x is the only vertex that v is distance 1 from, so v is distance 2 away only from those vertices connected to x. Since G is n-semiregular, v must be distance 2 away from n other vertices, so there must be n other vertices connected to x. Call them $y_1, y_2, y_3, \ldots, y_n$.

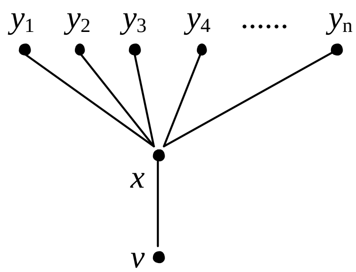


Figure 22

Now:

- v is distance 2 away from n other vertices ($y_1, y_2, y_3, \ldots, y_n$).
- Each y-vertex is distance 2 away from n other vertices (the other y-vertices and v).
- However, x is not distance 2 away from any other vertices.

We cannot connect v to any other vertices, so, in order to solve this problem, we have to connect new vertices onto one or more of the y-vertices.

Case 1: We connect the n new vertices to a single y-vertex.

Without loss of generality, say that the n new vertices are all connected to y_1. Call the new vertices $z_1, z_2, z_3, \ldots, z_n$. Now we have:

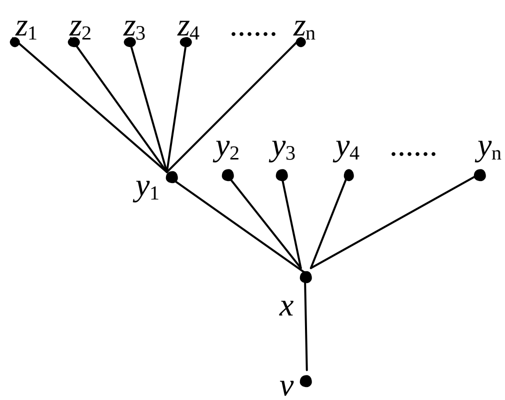


Figure 23

- v is distance 2 away from n other vertices ($y_1, y_2, y_3, \ldots, y_n$).
- x is distance 2 away from n other vertices ($z_1, z_2, z_3, \ldots, z_n$).
- Each y-vertex is distance 2 away from n other vertices (the other y-vertices and v).
- Each z-vertex is distance 2 away from n other vertices (the other z-vertices and x).

So this tree is n-semiregular. It is the n-barbell graph. We cannot connect any more vertices to the z-vertices, because that would make y_1 distance 2 away from more than n vertices.

Case 2: We connect the n new vertices to more than one of the y-vertices. Without loss of generality, assume that m of the new vertices are connected to y_1, and the remaining n-m of the new vertices are connected to other y-vertices.

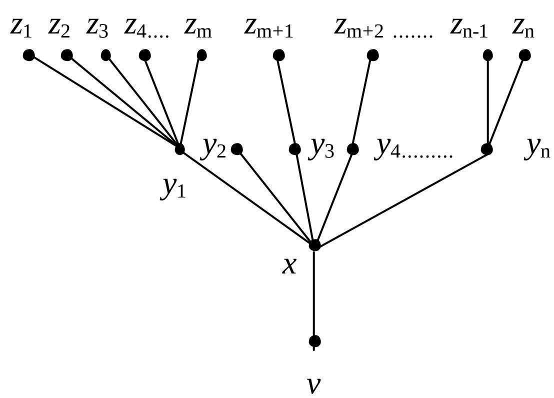


Figure 24

It can be seen that z_1 is now is distance 2 from only m other vertices, namely, $z_2, z_3, z_4, \ldots, z_m$ and x. Since $m<n$, our graph is not n-semiregular. We must see if it is possible to connect other vertices to z_1, making paths of length 2.

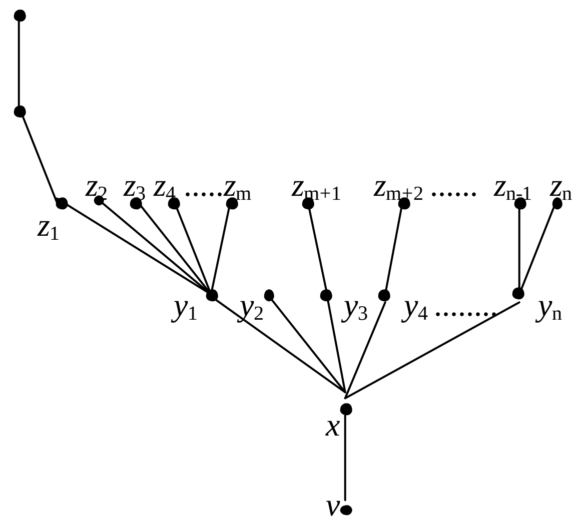


Figure 25

That does not work because it causes y_1 to be distance 2 from more than n other vertices. We have already stated that P_1 is a semiregular tree. We have seen that the barbell is semiregular, and the barbells are, by definition, trees. ✪

CHAPTER 4
MORE ON SEMIREGULAR GRAPHS

4.1. SYMMETRY

One way to think of the symmetries of a graph is by considering the number of automorphisms that the graph has. Every graph has at least one automorphism: the identity automorphism. Every automorphism aside from that could be considered a symmetry of the graph. We could then say that a graph is completely nonsymmetric if it has no automorphisms other than the identity automorphism.

All of the examples of semiregular graphs that we have seen until now have some symmetry. One might conjecture that every semiregular graph has some symmetry, but that is not the case. Consider the graph in Figure 26. The numbers within the faces of the graph represent the size of the cycles that determine the faces.

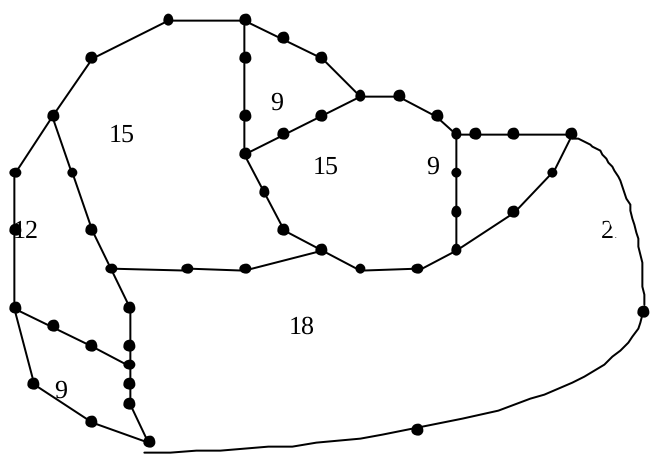


Figure 26

One can see that in any automorphism of this graph, the positions of the 18-cycle and the 12-cycle are fixed. That causes the 15-cycle and the 9-cycle that are adjacent to both of them to be fixed, and that in turn fixes the entire graph.

4.2. ADDING EDGES

It is an interesting property of semiregular graphs that in some cases a single edge can be added to an n-semiregular graph without changing the n-semiregularity of the graph. This property is never true with regular graphs. Figure 27 shows an example of this phenomenon. Both figures are 2-semiregular graphs.

Figure 27

Lemma 4.1. When an edge is added between two vertices a and b, the vertices that are distance 2 away from a given vertex in the graph will not change except possibly if that vertex is a, b, or a vertex adjacent to a or b.

Proof. Consider a connected graph G that contains vertices a and b, which are not connected by an edge. Now, let G' be exactly the same as G, but with an edge added between a and b. Let c be a vertex in G that is distance at least 2 away from a and from b. Any vertex that c is distance 2 away from in G will still be distance 2 from c in G', because the addition of an edge could not cause it to be any farther away, and because a distance less then 2 would be a distance of 1, but we did not add an edge to c. We must also check that c doesn't become distance 2 away from any vertex in G' that it was not already distance 2 away from in G. Assume that in G', c is distance 2 from a vertex that it was not distance 2 from in G. We shall call that vertex g. That means that there is a path c, f, g, for some vertex f in G'. Now, f cannot be equal to a or b, because we said that c was distance 2 from a and b. Since f is neither a nor b, the edges between c and f and between f and g must also exist in G. Therefore, g must be distance 2 from c in G as well.

Theorem 4.2. If G is an n-semiregular graph, and a and b are vertices in G that are not joined by an edge, then a and b can be connected without changing the n-semiregularity of G iff a is distance 2 away from every vertex connected to b, and b is distance 2 away from every vertex connected to a.

Proof. Let G be an n-semiregular graph, with a and b vertices in G that are not connected. Let a be distance 2 away from every vertex connected to b and b be distance 2 away from every vertex connected to a. If a and b are distance 2 away from each other, then there is a vertex c that is distance 1 from both a and b, which contradicts our assumption. Therefore, a and b are at least distance 3 apart. Now, consider connecting vertices a and b with an edge. Lemma 4.1 tells us that we need only concern ourselves with a, b and the vertices that are adjacent to a or b to determine whether the graph is still n-semiregular.

Consider vertex a. After the edge is added, any vertex originally connected to a remains distance 1 from a. Vertices a and b are more than distance 2 from each other in G, and distance 1 from each other after the edge is added. By adding the edge between a and b, a would become distance 2 away from any vertex adjacent to b. However, a is already distance 2 from every vertex connected to b. Thus, after the edge is added, a remains distance 2 away from exactly the same vertices as beforehand. Likewise, b remains at distance 2 from exactly the same vertices as before the edge was added.

Now consider g, a vertex adjacent to a. By Lemma 4.1, any vertex that is distance 2 or more from a and b is not affected by the addition of the edge. Vertex g is adjacent to a before and after the edge is added. By our assumptions, g is originally distance 2 from b, and remains so. In G, g is distance either 1 or 2 away from every other vertex that is adjacent to a. Since no edge is added to g, g remains the same distance from these vertices. Now let h be a vertex that is adjacent to b. If g and h are originally distance 1 apart, then they remain distance 1 apart. If they are originally distance 2 apart, then they remain distance 2 apart, because no edge was added between g and h. Assume that g and h were originally more than distance 2 apart, but that after the edge between a and b is added, they become distance 2 apart. That means that after the edge is added, there is a path g, j, h for some j in V(G). Vertex j cannot be either a or b, because by our assumptions, g must be distance 2 from b, and h must be distance 2 from a. Therefore the edges between g and j and between j and h must have existed in the original graph. Therefore, g and h must have always been distance 2 apart. So g remains distance 2 from exactly the same vertices as before the edge was added.

Say that for a connected n-semiregular graph G, a and b are two non-adjacent vertices in G with the property that adding an edge between a and b does not change the n-semiregularity of G.

Case 1: Say that a and b are distance 2 apart. After an edge is added between them, a will remain distance 2 away from all other vertices (besides b) that it was originally distance 2 away from (the distance cannot get longer, nor become 1). This means that unless a becomes, by the addition of the edge, distance 2 away from exactly one vertex that it was not already distance 2 away from, a will not be distance 2 away from n vertices, as it needs to be. When the edge is added between a and b, the only vertices that become distance 2 away from a that weren't originally distance 2 from a are the vertices that are connected to b but not to a and are not already distance 2 from a. This means that in G there must be exactly one vertex that is connected to b but is greater than distance 2 from a. Call that vertex g. Now any vertex from which g was originally distance 2 from, g must still be distance 2 from. However, g is now distance 2 from a, making g distance 2 from at least n +1 other vertices. This contradicts our assumption that G is an n-semiregular graph.

Case 2: Say that a and b are at distance more than 2 from each other. That means that there is no vertex connected to both a and b. Thus, when a and b are connected by an edge, a

would have to become (if it isn't already) distance 2 away from every vertex that is adjacent to b. Similarly, b would become connected to every vertex that is adjacent to a. Now, when the edge between a and b is added, a remains distance 2 from every vertex that it was originally distance 2 from. Therefore, in order to ensure that a is distance 2 away from only n other vertices, a must originally be distance 2 from every vertex adjacent to b. Similarly, b must originally be distance 2 from every vertex adjacent to a. ✪

4.3 CONNECTIONS BETWEEN REGULARITY AND SEMIREGULARITY

Another natural question to ask regarding semiregular graphs is whether there is any connection between regularity and semiregularity. The following is a method for transforming an n-semiregular into an n-regular graph, and visa versa.

Theorem 4.3. If G is an n-semiregular graph, let G^* be defined as the graph with the same vertex set as G, such that v_1 and v_2 are connected in G^* if and only if they are distance 2 away from each other in G. Then G^* is n-regular.

Proof. Let G be an n-semiregular graph. Let v be a vertex in G. v is then distance 2 away from exactly other vertices in G. Now consider v in G^*. In G^*, v is connected to exactly those vertices that it was distance 2 away from in G. That is, v is connected to exactly n other vertices. Since this is true for all vertices, G^* is n-regular. ✪

Figure 28 shows a graph G and the corresponding G^*.

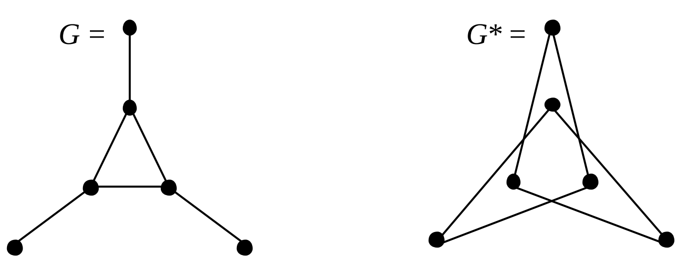


Figure 28

Theorem 4.4. If G is an n-regular graph, let G' is defined by inserting two vertices onto each edge of G. Then G' is an n-semiregular graph.

Figure 29 shows a graph G and the corresponding G'.

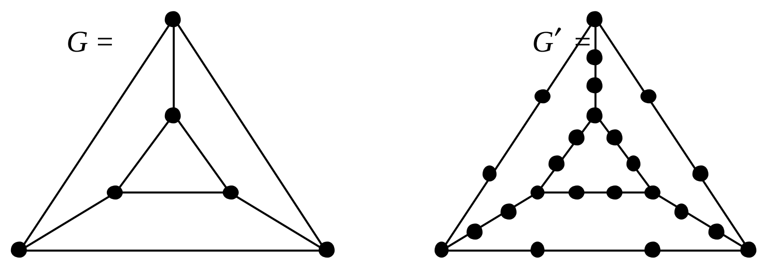


Figure 29

Proof. Let G be an n-regular graph, and G' as defined above. Let v be a vertex in G'. Then v may or may not have been a vertex in G.

Case 1: If v is a vertex of G, then in G vertex v was connected to exactly n other vertices:

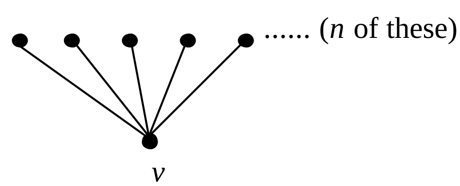


Figure 30

In G' we have:

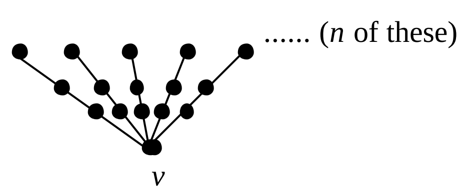


Figure 31

So v is distance 2 away from exactly n other vertices.

Case 2: If v is not a vertex of G, then v must have been added in along an edge of G. Say that v was added to the edge connecting v_1 to v_2 in G. Since G is n-regular, we have the following situation in G:

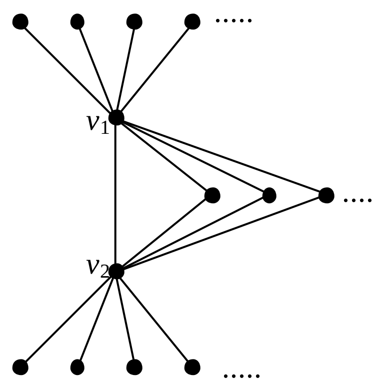


Figure 32

(Note that v_1 and v_2 may be connected to some of the same vertices.) Thus, in G' we have:

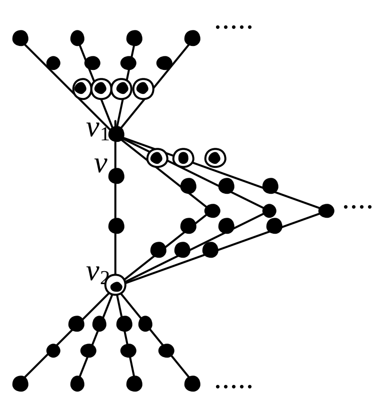


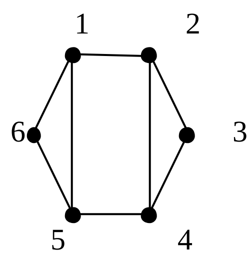
Figure 33

Vertex v is distance two away from exactly n other vertices; those which are highlighted with a double circle above.

Since $\deg_2(v)=n$ for every vertex v in G', G' is n-semiregular. ✪

4.4 ALGORITHM FOR DETERMINING SEMIREGULARITY

Once the vertices of a graph G have been labeled from 1 to n, the **adjacency matrix** of G is the $n \times n$ matrix where each element a_{ij} has value 1 if vertex i is connected to vertex j, and value 0 if vertex i is not connected to vertex j. Figure 25 shows an example of a graph and its corresponding adjacency matrix.



$$A = \begin{vmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

Figure 34

In order for the property of semiregularity to be determined by a computer, it is necessary to have an appropriate algorithm. The following algorithm will determine if a graph is n-semiregular for any n, given the adjacency matrix for the graph.

Algorithm 4.5. To compute whether G is n-semiregular for some n:

1. Start with A, the adjacency matrix of G.
2. Compute A^2.
3. Reduce all elements on the main diagonal to 0.
4. Change all elements with value greater than 1 to 1. Call the resulting matrix A^*.
5. Compute A^*-A.
6. If the number of positive 1's appearing in each row is the same for all rows, then the graph is semiregular. If each row contains exactly n positive 1's, then the graph is n-semiregular.

We shall first run through an example of this algorithm in use, and then prove the validity of the algorithm.

Example:

1. Start with adjacency matrix, A. (We use the same graph and adjacency matrix as in Figure 25 above.)

$$A = \begin{vmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

Figure 35

2. Compute A^2.

$$A^2 = \begin{vmatrix} 3 & 0 & 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & 0 & 1 \\ 1 & 2 & 1 & 0 & 3 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 \end{vmatrix}$$

Figure 36

3. Reduce all elements on the main diagonal to 0.

$$\begin{vmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{vmatrix}$$

Figure 37

4. Change all elements with value greater than 1 to 1. Call the resulting matrix A*.

$$\begin{vmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{vmatrix}$$

Figure 38

5. Compute A^*-A.

$$\begin{vmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{vmatrix}$$

Figure 39

6. We see that in the matrix above, each row contains exactly two positive 1's, so the graph in Figure 34 is 2-semiregular.

Proof. The proof of this algorithm relies on the fact that in the square of an adjacency matrix, the value of the element a_{ij} represents the number of distinct paths of length 2 between the vertices i and j.

Say that we have a graph G with adjacency matrix A. The pairs of vertices in G fall into four categories: 1) i and j are the same vertex 2) i are j are connected by an edge (i.e. there is a path of length 1 between i and j) 3) i and j are distance 2 apart 4) i and j are distinct vertices that are at least distance 3 apart. Now, for each of these possibilities, we shall consider the value of a_{ij} in A^*-A. For 1) we see that the value of a_{ij} in A is 0 and in A^* is 0 (since all the elements of the main diagonal were made to be 0). Thus the value of a_{ij} in A^*-A is 0. For 2), the value for a_{ij} in A is 1 and in A^* is either 0 or 1, so the value of a_{ij} in A^*-A is either 0 or -1. For 3), the value of a_{ij} in A is 0 and in A^* is 1, so a_{ij} =1 in A^*-A. For 4), the value of a_{ij} in both A and A^* is zero, so a_{ij}=0 in A^*-A, too. Thus we see that the value of a_{ij} in A^*-A is 1 if and only if i and j are distinct vertices that are distance 2 apart. Let n_i be the number of 1's appearing in the i^{th} row of the matrix. Then n_i represents the number of vertices that i is distance 2 away from. If $n_i = n$ for all i, then G is n-semiregular. ✪

APPENDIX A

2-SEMIREGULAR GRAPHS WITH ENDPOINTS

As stated in the proof of Theorem 3.4, we must do a case-by-case analysis to determine all the possible 2-semiregular graphs that have an endpoint. Before we begin, it will be useful to prove two lemmas.

Lemma A.1. If H is an induced subgraph of a 2-semiregular graph G, and $v \in V(H)$, and u_1, u_2,...., u_n are vertices of H that are adjacent to v such that $\deg_2(u_i) = 2$ in H, then if G includes any other vertex, u, connected to v, u is must also be connected to all the u_i.

Proof. Since H is an induced subgraph of G, if two vertices are distance 2 apart in H, then they are also distance 2 apart in G. Consider u_i. We know that $\deg_2(u_i) = 2$ in H. If u_i is not connected to u in G, then u_i is distance 2 from u, and $\deg_2(u_i) \geq 3$ in G. But G is a 2-semiregular graph, so u_i must be connected to u. ✪

This Lemma is helpful to speed up the process of evaluating all the cases. It also leads to another useful lemma.

Lemma A.2. If H is a 2-semiregular graph that includes an endpoint, and it is an induced subgraph of some connected 2-semiregular graph G such that there exists a vertex of G that does not belong to H, then G cannot contain an endpoint.

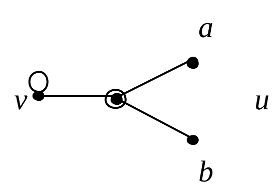
Proof. Let H be a 2-semiregular graph that has an endpoint, e. Let H be an induced subgraph of G such that G is a connected 2-semiregular graph with more vertices than H. Since G is connected, there must be a vertex v in G and not in H such that v is connected to one of the vertices of H. Since H is a 2-semiregular induced subgraph of G, Lemma A.2 tells us that v must be connected to all the vertices of H. But that means that v is connected to e, so e is not an endpoint in G. Furthermore, since all the vertices of G that are not in H are connected to all the vertices of H (and there has to be at least two vertices in H, since H is 2-semiregular) none of them can be endpoints. Thus, G does not contain an endpoint. ✪

Lemma A.2. shows us that once we have found a 2-semiregular graph with an endpoint, and we know all the connections between the vertices of that graph, we need not try to expand the graph further, since any resulting graph will not have an endpoint, and will therefore not concern us.

We must now show that the only 2-semiregular graphs that have an endpoint are graphs (a), (b), (c), (d), (e), (f), (g), (h), and (i) in Figure 19.

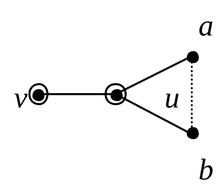
Proof. *Note*: In the following diagrams, ◉ will denote a vertex to which nothing more can be connected, and a dotted line between two vertices denotes that those two vertices *are not* connected.

Assume that G is a connected, 2-semiregular graph that has an endpoint, v. Then v must be connected to exactly one other vertex (the graph with one point is not 2-semiregular). Call it u. Furthermore, since G is 2-semiregular, v must be distance 2 away from exactly two other vertices. That means that u must be connected to exactly two other vertices, which we shall call a and b. Now consider what we have so far:



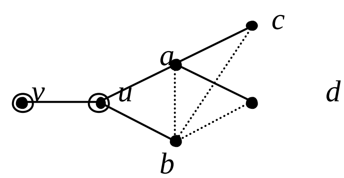
Now we must branch off into two separate cases: either a and b are connected by an edge, or they are not.

Case 1: Vertices a and b are not connected by an edge:

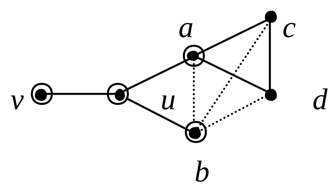


Vertex u must be distance 2 away from two other vertices, call them c and d. c and d cannot be connected to v, so they must be connected to a and/or b. There are four distinct ways in which this can be done.

Case 1a: Both c and d are connected to one of a or b, and not to the other. Without loss of generality, say that both c and d are connected to a.

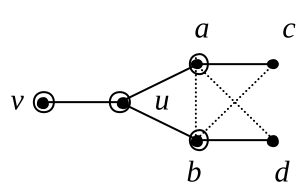


This is a 2-semiregular graph; the 2-barbell. It is (a) in Figure 19. If we assume that c and d are not connected, then we can apply Lemma A.2, and we are done. However, we must explore the possibility that c and d are connected:

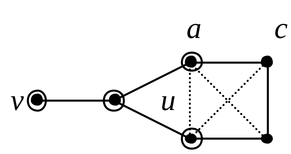


Now $\deg_2(c) = 1$. But we cannot add any new vertices onto c or d, because that would cause a to be distance 2 away from more than 2 vertices.

Case 1b: Vertex c is connected to a but not b, and vertex d is connected to b but not a.



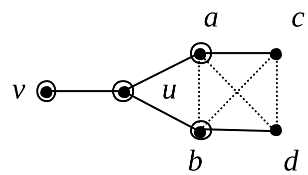
Now there are two cases: either c and d are connected, or they are not. If c and d are connected, then we get:



b *d*

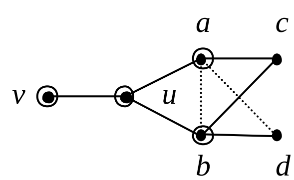
This is a problem, because now vertex *a* is distance 2 away from three other vertices.

If *c* and *d* are not connected, then we have:

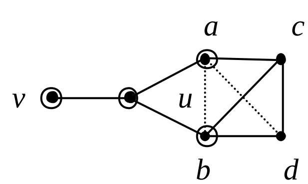


Now vertices *a* and *b* are already distance 2 away from two other vertices. Therefore no other vertices can be added onto *c* or *d*. The graph cannot be expanded, and it is not 2-semiregular, so this leads us nowhere.

Case 1c: Vertex *c* is connected to both *a* and *b*, but *d* is connected to *b* only:

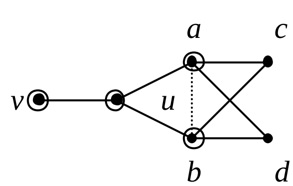


This is a 2-semiregular graph. It is (b) in Figure 19. Again, if we assume that *c* and *d* are not connected, then we can apply Lemma A.2 and we are done. However, we must try the case that *c* and *d* are connected by an edge:

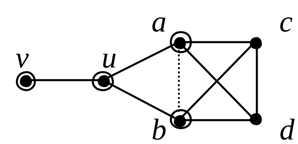


This case leads nowhere, because $\deg_2(a) = 3$.

Case 1d: Both vertices *c* and *d* are connected to both *a* and *b*:

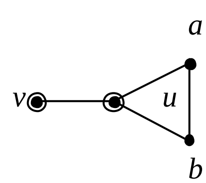


This is a 2-semiregular graph. It is (c) in Figure 19. But we must try the possibility that *c* and *d* are connected by an edge:



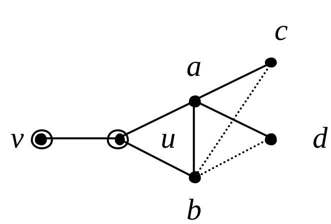
This case does not lead to anything because it is not 2-semiregular, and if any new vertices are connected onto *c* or *d*, then *a* will be distance 2 away from more than two other vertices.

Case 2: Vertices a and b are connected by an edge.



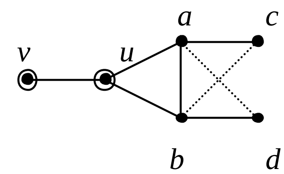
Since *G* is 2-semiregular, vertex *u* must be distance 2 away from exactly two other vertices. Just as in Case 1, the vertices that *u* is distance 2 away from (*c* and *d)* must be connected to *a* and/or *b*, and there are four distinct ways in which this can be done:

Case 2a. Both *c* and *d* are connected to one of *a* or *b*, but not to the other. Without loss of generality, say that both *c* and *d* are connected to *a*, and not to *b*.

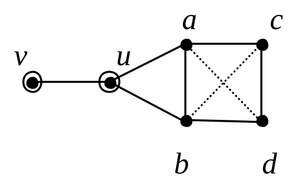


This causes *b* to be distance 2 away from three other vertices: *v*, *c*, and *d*. Thus, *G* is not 2-semiregular.

Case 2b. Vertex c is connected to a but not b, and vertex d is connected to b but not a.

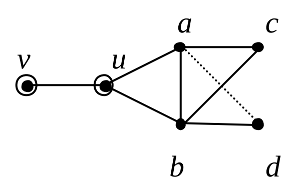


This is a 2-semiregular graph. It is (d) in Figure 19. If we assume that c and d are not connected, then Lemma A.2 tells us that we are finished. However, we must try connecting vertices c and d with an edge. We get:

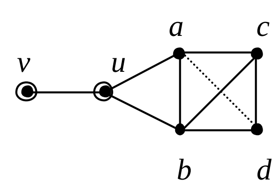


This graph is also 2-semiregular. It is (e) in Figure 19.

Case 2c. Vertex c is connected to both a and b, but vertex d is connected only to b.

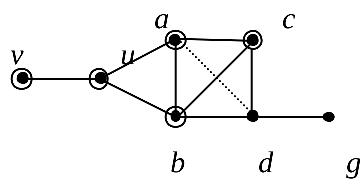


We see that vertex d is now distance 2 away from three other vertices (u, a, c). We must then shorten the path between d and one of the vertices that d is distance 2 away from. We know that d cannot be connected to a or u, so we must add an edge between d and c.



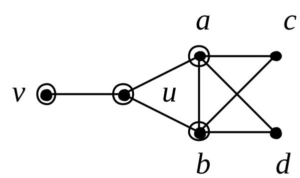
No other vertices can be connected to a and b. Neither can a new vertex be connected to c because that would make vertex a distance 2 away from more than 2 vertices.

However, vertex b is distance 2 away from only one other vertex. We must add something; the only possibility is connecting vertex d to a new vertex, g.



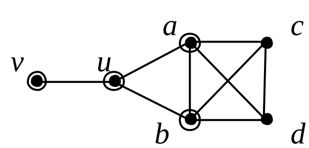
This graph is 2-semiregular; it is (f) in Figure 19. We know from Lemma A.2 that we cannot connect d or g to any new vertices.

Case 2d. Both vertices c and d are connected to both a and b.

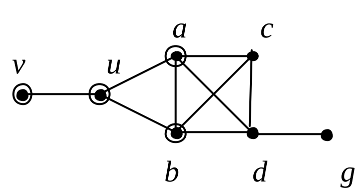


No new vertices could be connected onto a or b, because that would cause vertex u to be distance 2 away from more than 2 other vertices. We must try the two cases of when c and d are and are not connected.

Case 2d(1): If c and d are connected:

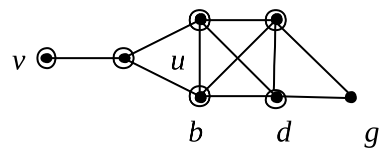


The only vertices that can be connected to a new vertex are c and d. Without loss of generality, say that d is connected to a new vertex, g.

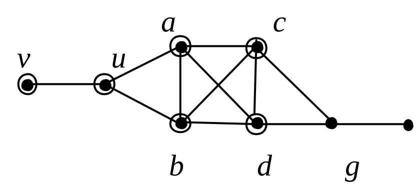


Vertex g is distance 2 away from a, b and c. We must then connect g to c:

a c

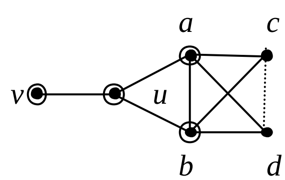


This graph is still not 2-semiregular, and the only vertex to which it is possible to connect a new vertex is g, so we try that:

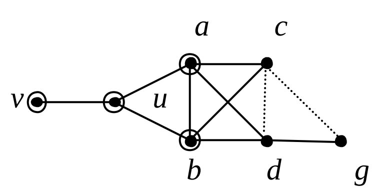


This graph is 2-semiregular. It is (g) in Figure 19. We cannot add any more vertices.

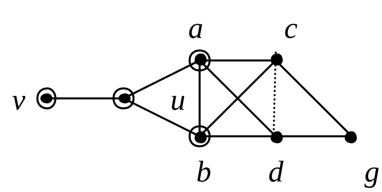
Case 2d(b): Vertices c and d are not connected.



Vertices c and d are the only ones that could be connected to a new vertex. Without loss of generality, say that a new vertex, g, is connected to c. Vertex g may or may not be connected to d. If d and g are not connected:



This graph is 2 semiregular. It is (h) in Figure 19. No new vertices can be connected to c, d, or g. If d and g are connected:



This is a 2-semiregular graph. It is (i) in Figure 19. No new vertices can be connected to c, d, or g.

Thus, the only possible connected 2-semiregular graphs that have an endpoint are graphs (a), (b), (c), (d), (e), (f), (g), (h), and (i) in Figure 19.

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BIOGRAPHICAL SKETCH

Alison Pechin Northup was born in Lansing, Michigan on December 10, 1979. Her family relocated to Fort Myers, Florida in 1981, where Alison attended Cypress Lake Center for the Arts, a Magnet high school. After graduating from High School in 1998, she enrolled at Stetson University.

Alison was involved in Math Team from seventh through twelfth grade. In college, she has participated in the Putnam Examination and the Carleton-St. Olaf Summer Mathematics Program for women. She is currently pursuing a Bachelor of Science in Mathematics with a minor in Spanish.