Mancala is an African game with myriad variations. Many historians believe it to be the oldest game in the world. Thw word Mancala means "to transfer" in Arabic. Variations of Mancala include ti, kpo, wari, azigo, igisoro, and omweso. Mancala is played on a board with six cups on either side and a base at each end. The bases are called kalahas. The cups are filled with a certain number of stones the value of which depends on the variation being played. A player moves by taking all of the stones from one of his/her cups and placing, or sowing, the stones one at a time in adjacent cups until all of the stones have been moved. In some variations of the game, if the last stone from a cup is placed in the player's respective base (i.e., the base towards which the player is moving his/her stones), the player is granted a bonus move and is allowed to move again. Otherwise, the player's turn ends. If stones remain after a stone has been placed in the base, the remaining stones are sown according to the definition of sowing on the opponent's side of the board. The player that removes all of the stones from his side of the board wins the game. We refer to cearing all of the stones off one side of the board as solving the game. Through our investigation, we use general patterns to determine mathematical formulas which enable us to predict the outcome of any game based on the initial positions of the stones as well as determine the best strategy for winning the game.

In order to analyze this version of Mancala, we ignore the second player completely and concentrate solely on determining the number of moves required to clear all of the stones off a side depending on the stone placements, i.e., the positions containing stones and the number of stones in each of those positions. We also extend the board to include an infinite number of cups so as to facilitate the discovery of general formulas. In order to standardize our analysis, we number the cups from the base as positions 1,2,3, etc.

We will begin by constructing all the ston placements for the first m cups (where the mth cup is non-empty) which can be solved in one move. We let u(n) be the number of stones initially in the nth cup in such a configuration. We let v(n) be the number of moves made from the nth position. We will recursively define these values of u(n) and v(n) by using the values for these variables already determined for positions greater than n. We let w(n) be the sum of v(i)'s for all i's from (n+1) to m. Hence w(n) is the number of moves made from positions greater than n.

Theorem 1: u(n)=(n-w(n)) mod n.

Proof: Each move made from a position greater than n adds one stone to the nth cup. Thus, the number of stones in the nth cup at any given time is u(n) plus the number of moves made from positions greater than n up to that time. By definition, in order to make a bonus move, the number of stones in the nth cup must equal n. Thus, in order to clear all of the stones on the board, u(n)+w(n) must equal a multiple of n. Therefore, u (n)-(n-w(n)) mod n.

If u(n)=0, either zero or n may be used. while these two alternatives lead to two different stone placements, both placements can be solved in one move. The mth cup must contain m stones since it must be non-empty and w(m)=0. Any number of stones may be used in the first cup since all possible values are equivalent to zero mod 1. This will be proved as Proposition 2.

Theorem 2: v(n)=the quotient of (u(n)+w(n)) mod n=(u(n)+w(n))/n.

Proof: Each time the number of stones in the nth cup equals n, one move is made from this cup. Since u(n) +w(n) equals the maximum number of stones in the nth cup, v(n) equal the quotient of (u(n)+w(n)) mod n which is the number of times the nth cup contains n stones. Since u(n)+w(n) must equal a multiple of n in order to clear all of the stones off of the board, v(n) must also equal (u(n)+w(n))/n. Since the mth cup contains m stones and w(m)=0, only one move is made from that position. Thus v(m)=1.

To illustrate these concepts, we will demonstrate how these theorems are applied to the case where m=7. We know that the seventh cup must contain 7 stones. Thus, u(7)=7. We also know that v(7)=1. Using these valued, we see that u(6)=(6-v(6)) mod 6. Since the one move made from the seventh position is the only one made from a position greater than 6, w(6)=1. Thus, u(6)=(6-1) mod 6=5. We also see that v(6)=(5+1)/6=1. This additional move implies that w(5)=2. Thus, u(5)=(5-2) mod 5=3 and v(5)=3+2/5=1. Hence w(4)=3. u(4)=(4-3) mod 4=1 and v(4)=(1+3)/4=1. This implied that w(3)=4. Thus, u(3)=(3-4) mod 3=2 and v(3)=(2+4)/3=2. Thereforew(2)=6. u(2)=(2-6) mode 2=0. In this case, either zero or two can be used. If zero is used, v(2)=(0+6)/2=3. If 2 is used, v(2)=(2+6)/2=4. The variable u(1) can have any value as explained above.

This is more easily seen using a chart (Table 1) showing the moves made from each position for the case involving seven cups. Underlined valued represent a move made from any particular position.

We now examine some propositions together with their proofs.

Proposition 1: If only one bean remains in the nth cup where the nth cup is the only non-empty position, n moves are required to clear the board.

Proof: Since only one stone is on the board, there is only one move which can be made. The stone can only be moved from position n to position (n-1). Since the base corresponds to position 0, only n such moves are necessary to remove the stone from the board.

Table 1

Position Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**
u(i)	0*	0	2	1	3	5	$\underline{7}$
		1	1	$\underline{3}$	2	4	6
		$\underline{2}$	2	0	2	4	6
		3	0	0	2	4	$\underline{6}$
		4	1	1	3	$\underline{5}$	
		5	$\underline{2}$	2	4		
		6	0	2	$\underline{4}$		
		7	1	$\underline{3}$			
		8	$\underline{2}$				
		$\underline{9}$					
v(i)	*	3	2	1	1	1	1

* As previously explained, the number of stones in the first position, and therefore the number of moves made from the first position, does not affect any of the values needed to make any of the calculations for the other positions.

Proposition 2: Unless the first cup is the only cup containing stones, the number of stones in the first position does not affect the number of moves required to clear the board.

Proof: If the first position is the only one containing stones, then only one move is possible and necessary to remove the stones. If the first cup is also empty, then the board is already clear. Thus, in this case, the stones in the first cup cannot be completely ignored without affecting the number of moves to clear the board. Now, we examine the case where the first position is not the only non-empty one. By the definition of sowin, the first position receives an additional stone for each move made from positions greater than n. Thus, only a move made from the first position can clear the board. Since only one move is required to clear the first cup regardless of the number os stones contained in it, the number of stones in the first cup does not affect the number of moves required to clear the board. Thus, the number of stones initially in the first cup can be ignored.

We will now determine the number of moves necessary to clear the board when there are only two stones on the board (see Table 2). The rows and columns represent the placements of the two stones and the values in the table are the number of moves required to clear the stones off the board.

In order to determine the valued for each space on the data table, we let f(x,y) represent the minimum number of moves required to clear the board where x,y represent positions containing stones (x<=y).

Table 2

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	1	2	3	4	5
3	3	2	3	4	5	6
4	4	3	4	3	4	5
5	5	4	5	4	5	6
6	6	5	6	5	6	5

Theorem 3:

$$f(x,y) = \begin{cases} y & x \text{ odd} \\ y-1 & x \text{ even} \end{cases}$$

Proof: To prove this theorem, we verify that f holds for small values of x and y. We have already shown that only one move is necessary to clear the board when there are two stones in the first cup. We now examine the case where one stone is in the nth position and the other is in the first position. By Proposition 2, we know that we can treat this case as if there was only the one stone in the nth cup. By Proposition 1, we know that n moves are required to clear the board in this case. This allows us to complete the first row and column of the table.

Note that the diagonal spaces represent the cases where both stones are in the same cup. By the definition of sowing, one of the stones is moved two spaces and the other is moved only one space. This is equivalent to a knight move on the data table. That is, after one move is made from a diagonal space, the new stone placements is represented in the space two rows above and one column to the left of the original space. Since the data table is symmetric about the diagonal, this would be the same as moving one row above and two columns to the left. For nondiagonal spaces, only one of the stones is moved one space. This means that a move froma non-diagonal space results in a new position that is one space to the left or above the original position.

We now assume that f is valid for all values of x,y where $x+y<=n$ and prove that it is valid for $x+y=n+1$ by showing that $f(x,y)$ is the minimum value for all possible moves from $(x,y)+1$. First, we let $x=y$. This represents spaces along the main diagonal of the table. As previously described, knight moves are made from these spaces. Thus, we need to show

$$f(x,x)=f(x-2,x-1)+1$$

since one move is required to move the two rows up and one column to the left. If x is odd, then x-2 is odd and $f(x,x)=x=(x-1)+1=f(x-2,x-1)+1$. Now, we let $x<y$. This refers to all spaces not on the main diagonal. As seen above, moves made from these spaces move one space either above or to the left of the original space. Thus $f(x,y)$ must equal $\min(f(x-1,y),f(x,y-1))+1$ in order to be consistent for all values of x,y. We show this algebraically.

$$\begin{array}{ll} & (\{y \quad x-1\ \text{odd} \quad \{\ y-2\ \ x\ \text{even} \\ f(x,y)=1+\min & (\{ \\ & (\{y-1\ \ x-1\ \text{even}\ \ \{\ y-1\ \ x\ \text{odd} \end{array}$$

Since, if x is even, x-1 is odd and if x is odd, x-1 is even,

$$\begin{array}{ll} & ((y,y-2) \qquad x\ \text{even} \\ f(x,y)=1+\min & (\\ & ((y-1\ ,y-1)\ \ x\ \text{odd} \end{array}$$

Thus, if x is odd, $f(x,y)=1+(y-1)=y$. If x is even, $f9x,y)=1+(y-2)=y-1$. This shows that the formula $f(x,y)$ is consistent for all values of x,y.

Note that since we are ignoring the possibility that stones left over after a stone is sown in the base can be sown on the opponents side of the board, we assume that any extra stones remain in the base. however, the values of the positions of these stones is essential for the calculation of $f(x,y)$ values. Thus, we assign the base position the value of 0 and each additional stone the value -1,-2,-3, etc.

We now examine the case of the stone placements containing three stones. We assign these stones the values x,y, and z where $x<=y<=z$. We define $g(x,y,z)$ as we defined $f(x,y)$ using the properties of x,y,z defined above.

Theorem 4:

$$g(x,y,z)=\begin{cases} z & x<=2,\ x+y \text{ even} \\ z-1 & \text{otherwise} \end{cases}$$

Proof: As in the case of f, g(x,y,z) represents the minimum number of moves required to clear the stones off the board. To determine if this theorem holds, we use a proof similar to that which we use to prove Theorem 3. We first show that g is consistent for small values of x,y, and z. We then assume that g is valid for all values x,y,z where x+y+z<=n and then prove that g is valid for x+y+z=n+1 by showing that g(x,y,z) equals the minimum value for all moves from (x,y,z) plus 1.

We first show that g is valid for small values of x,y,z.

Proposition 3: For the three stone case, if one of the three stones is in the first position, it can be treated as a two stone case. If two stones are in the first cup, it can be treated as a one stone case.

Proof: If we let x=y=1, by Proposition 1 we know that this is equivalent to the one stone case. We know that for this case, if the stone is in the zth position, k=z which is consistent with the definition of g. We next look at the case where x=1. Again by Proposition 1, we know that this case is equivalent to the two stone case. If y is odd, f(y,z)=z=g(1,y,z). If y is even f(y,z)=z-1=g(1,y,z).

We now let x=y=z=3. Since the only move from this placement rewards the player with a bonus move and results in the (1,2) placement. Thus, g(3,3,3)=2=f(1,2). Now let x=y=2. Since a move from position 2 results in a bonus move and the placement (1,z) which is the same as the one stone case where z is the only non-empty position. Thus, g(2,2,z)=z. If a move is made from the z position, g(2,2,z)=1+g(2,2,z-1)=1+(z-1) =z. Hence, we see that g is valid for all of these small values of x,y,z.

We now examine larger values of x,y,z. We find that there are four cases within this three stone case.

Case 1: x=y=z

This case corresponds to the stone placement where all three stones are in the same cup. Only one move can be made from this placement. This move results in the placement (x-3,x-2,x-1). Thus, we must show that

$$g(x,x,x)=1+g(x-3,x-2,x-1)$$

Case 2: x<y=z

This case corresponds to the stone placement where the stone in the least position is alone and the other two stones are together in the same position with a greater value. Two moves can be made from this placement. One move transfers the stones from the position (x,y,y) where x<y to the position (x-1,y,y). The other move transfers the stones from the same initial position to the position (x,y-2,y-1). This move is similar to the knight move encountered in the two stone case. We must show that

$$g(x,y,y)=1+\min(g(x-1,y,y),g(x,y-2,y-1)).$$

We first examine the case where $x+y$ is even and $x<=2$. Under these conditions, $g(x-1,y,y)=y-1$ since $(x-1)+y$ is not even and $g(x,y-2,y-1)=y-1$ since $x+(y-2)$ is even. Thus, $g(x,y,y)=1+(y-1)=y$ which is consistent with the formula. We now examine the otherwise case. For this case, $g(x,y-2,y-1)=y-2$ since either $x+(y-2)$ is not odd or $x>2$. The other move is more complicated. If $x+y$ is even, then $(x-1)+y$ is not even and $g(x-1,y,y)=y-1$. If $x+y$ is not even and $x>3$, then $g(x-1,y,y)=y-1$. However, if $x+y$ is not even and $x=3$, $g(x-1,y,y)=y$ since $(x-1)+y$ is even and $x-1=2$. This case, however, does not affect the calculation of $g(x,y,y)$ since

$$\min(g(x-1,y,y),g(x,y-2,y-1))=y-2$$

Thus $g(x,y,y)=1+(y-2)=y-1$ which is also consistent with Theorem 4.

Within this case, we find another interesting situation if $x=y-1=z-1$. For this special case, we must show that

$$g(x,x+1.x+1=1+\min(g(x-1,x+1,x+1),g(x,x,x-1))$$

since these are the only two moves possible from this placement. We rearrange the last placement so that the positions increase numerically. Now we examine the case where $x<=2$ and $x+y$ is even. Under these conditions, $g(x-1,x+1,x+1)=x+1$. Since $(x-1)+x$ is not even, $g(x-1,x,x)=x-1$. Thus, $g(x,x+1,x+1=1+(x-1)=x$ which is consistent with the formula. This case, however is irrelevant since $x+(x+1)$ can never be even. Now, if $x<=3$, $g(x-1,x+1,x+1)=x+1$ since $(x-1)+(x+1)$ is even. If $x>3$. $g(x-1,x+1,x+1)=x$. $g(x-1,x,x)=x-1$ always since $(x-1)+x$ is never even. Thus, $g(x,x+1,x+1)=1+(x-1)=x$.

Case 3: $x=y<z$

This case corresponds to the placement where two stones are together in the position with the least value and the other stone is alone in a position of greater value. As in Case 2, two moves are possible from this placement. One move transfers the stones from the placement (x,y,z) to the placement $(x-2,x,z-1)$. We must show that $g(x,x,z)=1+\min(g(x-2,x-1,z),g(x,x,z-1))$. We first examine the case where $x<=2$. Since $x=y$, $x+y$ is always even. Since $(x-2)+(x-1)$ is not even, $g(x-2,x-1,z)=z-1$. $g(x,x,z-1)=z-1$ also. Thus, $g(x,x,z)=1+(z-1)=z$. Now, we examine the case where $x>3$. We see that $g(x-2,x-1,z)=z-1$. Since $(x-2)+(x-1)$ is never even, we do not find the same special case as in Case 2. On the other hand, we see that $g(x,xx,z-1)=z-2$. Thus, $g(x,x,z)=1+\min(z-2,z-1)=1+(z-2)=z-1$.

Case 4: $x<y<z$

This case corresponds to the placement where all three stones are in separate positions. Three moves are possible from this placement. The stones can be transferred from the initial placement to $(x-1,y,z)$,$(x,y-1,z)$, or $(x,y,z-1)$. Thus, we must show that

$$g(x,y,z)=1+\min(g(x-1,y,z),g(x,y-1,z),g(x,y,z-1))$$

As before, we examine the case where $x<=2$ and $x+y$ is even. Since $(x-1)+y$ is not even, $g(x-1,y,z)=z-1$. Similarly, since $x+(y-1)$ is not even, $g(x,y-1,z)=z-1$. $g(x,y,z-1)=z-1$ also. Thus, $g(x,y,z)=1+(z-1)=z$ which is consistent with the formula. Now we examine the otherwise case. If $x<=3$ and $x+y$ is not even, $g(x-1,y,z)=z$. Otherwise, $g(x-1,y,z)=z-1$. If $x<=2$ and $x+y$ is not even, $g(x,y-1,z)=z$. Otherwise, $g(x,y-1,z)=z-1$. As long as $x>2$ or $x+y$ is not even, $g(x,y,z-1)=z-2$. Thus, the minimum value for the otherwise case is $z-2$ and $g(x,y,z)$

=1+(z-2)=z-1.

In order to more easily recognize the relationships between these three stone placements and their corresponding g(x,y,z) values, we construct the following data tables based on several z valued (see Table 3).

We have categorized stone placements that can be solved in one move (see Theorems 1,2) and found the minimum number of moves required to solve all positions with no more than three stones (see Theorems 3,4). Note that in the cases above, the minimum number of moves required to clear the board can always be achieved by moving the stones in the position with the greatest value. Also note that moving stones from any position which results in a bonus move either does not affect this minimum value for the stone placement or lowers it. This observation leads us to the following corollary:

Corollary: The best strategy for all placements with no more than three stones or placements that can be solved in one move is to do the following: 1) first remove the stones form any position that results in a bonus move (moving first from cups closest to the base), and 2) then move the stones from the position with the greatest value.

By generalizing these results, we make this conjecture:

Conjecture: For all possible stone placements, the best strategy to solve the game is to first remove the stones from any positions that result in a bonus move (proceeding according to increasing position value as described in the corollary) and then move the stones from the position with the greatest value.

Table 3

z=1

	1
1	1

z=2

	1	**2**
1	2	1
2	2	2

z=3

	1	**2**	**3**
1	3	2	3
2	2	3	2
3	3	2	2

z=4

	1	**2**	**3**	**4**
1	4	3	4	3
2	3	4	3	4
3	4	3	3	3
4	3	4	3	3

z=5					
	1	**2**	**3**	**4**	**5**
1	5	4	5	4	5
2	4	5	4	5	4
3	5	4	4	4	4
4	4	5	4	4	4
5	5	4	4	4	4

z=6						
	1	**2**	**3**	**4**	**5**	**6**
1	6	5	6	5	6	5
2	5	6	5	6	5	6
3	6	5	5	5	5	5
4	5	6	5	5	5	5
5	6	5	5	5	5	5
6	5	6	5	5	5	5