Introduction

It is the purpose of this paper to explore a variant of Pascal's triangle. This variant has the rule that every entry, denoted as $a_{n,k}$, where $a_{1,1}$=1, is calculated in a way such that

$a_{n,k} = \sum_{i=1}^{k-1} a_{n-i,k-i} + \sum_{i=k}^{n-1} a_{i,k}$. This means that when these numbers are put into a triangular formation, every number is the sum of all the numbers above it in its two diagonals. The beginning of the triangle looks like the following:

```
          1
        1   1
      2   2   2
    4   5   5   4
  8  12  14  12   8
16  28  37  37  28  16
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For example, 14 = 2 + 5 + 5 + 2. Now, it is important to understand some notation and terminology before going any further. The ***rows*** of the triangle will begin at n=1 and work down the triangle in increments of 1. The ***columns*** of the triangle point southwest (60^{o} from horizontal) and will begin at k=1, and proceed in increments of 1. For example, we would say that the leftmost number 12 would be in row 5 and in column 2. ***Shallow diagonals*** point southwest (30^{o} from horizontal). For example, the third shallow diagonal consists of the elements $a_{3,1}$ and $a_{2,2}$. The sum of the elements in any row n will be represented by S_n. The sum of the elements in any shallow diagonal n will be represented by D_n. Finally, ***anticolumns*** are the columns of the triangle pointing southeast that begin at k`=1, and proceed in increments of 1 moving diagonally from right to left. So, for example, the rightmost number 12 would be in row 5 and in anticolumn 2. Now that the notation is defined, we will explore the new triangle we have defined.

Results

Some interesting discoveries found within this triangle will now be discussed, with some proven, in the order that they were found.

Lemma 1: $a_{n,1} = 2a_{n-1,1}$ for n$\geq$3.

Proof: By definition, $a_{n,1} = \sum_{i=1}^{n-1} a_{i,1} = a_{n-1,1} + \sum_{i=1}^{n-2} a_{i,1} = a_{n-1,1} + a_{n-1,1}$, since $a_{n-1,1} = \sum_{i=1}^{n-2} a_{i,1}$ by definition. Hence, $a_{n,1} = 2a_{n-1,1}$ for n$\geq$3.

Theorem 1: $a_{n,1} = 2^{n-2}$ for n$\geq$2.

Proof: Since $a_{n,1} = 2a_{n-1,1}$, then the theory of difference equations implies $a_{n,1} = \text{C } 2^{n-1}$.

Solving for C when $a_{4,1} = 4$. We get C = ½ which means $a_{n,1} = 2^{n-2}$ for n$\geq$2.

Lemma 2: $a_{n+1,2} = 2a_{n,2} + 2^{n-3}$ for n$\geq$3.

Proof: We know $a_{n+1,2} = a_{n,1} + \sum_{i=2}^{n} a_{i,2}$ by definition. Further, $a_{n,1} = a_{n-1,1} + \sum_{i=1}^{n-2} a_{i,1}$ and $\sum_{i=2}^{n} a_{i,2}$ $= a_{n,2} + \sum_{i=2}^{n-1} a_{i,2}$. Thus, $a_{n+1,2} = a_{n-1,1} + a_{n,2} + \sum_{i=1}^{n-2} a_{i,1} + \sum_{i=2}^{n-1} a_{i,2}$. But, $\sum_{i=1}^{n-2} a_{i,1} = a_{n-1,1}$. So, $\sum_{i=1}^{n-2} a_{i,1} + \sum_{i=2}^{n-1} a_{i,2} = a_{n,2}$ by definition, which implies $a_{n+1,2} = a_{n-1,1} + 2a_{n,2}$. Therefore, by Theorem 1, $a_{n+1,2} = 2a_{n,2} + 2^{n-3}$ for n$\geq$3.

Theorem 2: $a_{n,2}$ = (n+1) 2^{n-4} for n≥4.

Proof 1: The theory of difference equations implies $a_{n,2}$ = (A + Bn)(2^{n-4}). Solving for A and B when $a_{3,2}$ = 2 and $a_{4,2}$ =5, we get A=B=1. So, $a_{n,2}$ = (n+1) 2^{n-4}.

Proof 2: (By Induction) When n = 4, $a_{4,2}$ = 2 (2) + 2^0 = 5 = (4+1) 2^0. Now assume that $a_{n,2}$ = (n+1) 2^{n-4} is true for some n. Since $a_{n+1,2} = 2a_{n,2} + 2^{n-3}$ we have $a_{n+1,2}$ = 2((n+1) 2^{n-4}) + 2^{n-3} = (n+2) 2^{n-3}, and we conclude by induction that $a_{n,2}$ = (n+1) 2^{n-4} for n≥4.

Lemma 3: $a_{n,3} = 2a_{n-1,3} + a_{n-2,2} + a_{n-2,1}$ for n≥4.

Proof: By definition, $a_{n,3} = \sum_{i=1}^{2} a_{n-i,3-i} + \sum_{i=3}^{n-1} a_{i,3} = a_{n-1,2} + a_{n-2,1} + \sum_{i=3}^{n-1} a_{i,3} = 2a_{n-2,2} + a_{n-3,1} + a_{n-2,1} + a_{n-1,3} + \sum_{i=3}^{n-2} a_{i,3} = a_{n-2,2} + a_{n-2,1} + a_{n-1,3} + a_{n-3,1} + a_{n-2,2} + \sum_{i=3}^{n-2} a_{i,3}$. But, $a_{n-1,3} = \sum_{i=1}^{2} a_{n-i,3-i} + \sum_{i=3}^{n-2} a_{i,3} = a_{n-3,1} + a_{n-2,2} + \sum_{i=3}^{n-2} a_{i,3}$ by definition. Thus, $a_{n,3} = a_{n-2,2} + a_{n-2,1} + a_{n-1,3} + a_{n-1,3} = 2a_{n-1,3} + a_{n-2,2} + a_{n-2,1}$ for n≥4.

Theorem 3: $a_{n,3} = (-4 + 7n + n^2)2^{n-7}$ for n≥4.

Proof 1: The theory of difference equations implies $a_{n,3}$ = C(2^{n-7}) + Dn(2^{n-7}) + En^2(2^{n-7}). Solving for C, D, and E when $a_{4,3}$ = 5, $a_{5,3}$ = 14, and $a_{6,3}$ = 37, we get C= -4, D= 7, and E= 1, which gives $a_{n,3} = (-4 + 7n + n^2)2^{n-7}$.

Proof 2: (By Induction) First, for n=5, $a_{5,3} = 2a_{4,3} + a_{3,2} + a_{3,1} = 2(5) + 2 + 2 = 14 = (-4 + 7(5) + 5^2)\,2^{-2}$. Now assume that $a_{n,3} = (-4+7n+n^2)2^{n-7}$ is true for some n. Proceeding inductively, $a_{n+1,3} = 2a_{n,3} + a_{n-1,2} + a_{n-1,1} \Rightarrow a_{n+1,3} = (-4+7n+n^2)2^{n-6} + n2^{n-5} + 2^{n-3} \Rightarrow$ $a_{n+1,3} = 2^{n-3}[(-4+7n+n^2)2^{-3} + n2^{-2} + 1] \Rightarrow 2^{n-6}\,2^3\,[(1/2) + (9n/8) + (n^2/8)] \Rightarrow (4 + 9n + n^2)$ 2^{n-6}, and we conclude by induction that $a_{n,3} = (-4+7n+n^2)2^{n-7}$ for $n \geq 4$.

Lemma 4: $S_n = 3S_{n-1}$ for $n \geq 3$.

Proof: It can be shown that $S_n = 2S_{n-1} + S_{n-1}$, and since $S_{n-1} = 2S_{n-2} + 2S_{n-3} + \ldots + 2S_1$. This means $S_n = 3S_{n-1}$.

Theorem 4: $S_n = (2/9)\,3^n$ for $n \geq 2$ where $S_1 = 1$ and $S_2 = 2$.

Proof 1: From the theory of difference equations and lemma 4, $S_n = C + D3^n$. Since $S_2 = 2$ and $S_3 = 6$, solving for C and D, we get C = 0 and D = (2/9). Thus, $S_n = (2/9)\,3^n$.

Proof 2: (By Induction) First, for n=3, $S_3 = 3S_2 = 3(2) = 6 = (2/9)\,3^3$. Now assume that $S_n = (2/9)\,3^n$ for some n. So, $S_{n+1} = 3S_n \Rightarrow 3(2/9)\,3^n \Rightarrow S_{n+1} = (2/9)\,3^{n+1}$. Thus, by induction, $S_{n+1} = (2/9)\,3^{n+1}$ for $n \geq 2$.

Conjecture 1: The triangle is symmetrical (i.e.: $a_{n,k} = a_{n,n+1-k}$**).**

Conjecture 2: $S_n = 2S_{n-1} + 2S_{n-2} + 2S_{n-3} + \ldots + 2S_1$ for $n \geq 2$.

Conjecture 3: $D_n/3 = D_{n-1}/3 + D_{n-2}$ for $n \geq 6$.

Conjecture 4: The only odd numbers in the triangle are $a_{1,1}$ and entries of the form $a_{2n,k}$ and $a_{2n,k`}$. These are consecutive numbers in the middle of every even numbered row.

Conjecture 5: If $a_{n,k} \equiv a_{n,k+1}$ (mod 3), then $a_{n+1,k+1} \equiv a_{n,k}$ (mod 3).

Plans for Next Semester

There are some interesting patterns in the triangle (mod 3), besides Lemma 1, which means that next semester, these patterns will be further explored. In addition, there does not seem to be any interesting patterns emerging from the triangle (mod n for other n). A goal for next semester is to have all of the conjectures proven. Also, I have just received an article titled, "A New Look at Fibonacci Generalization" from the library through interlibrary loan. This is in reference to conjecture 2, and may help in finding a proof for it, as well as some other interesting information. Another goal for next semester is to find out how, if possible, any of the results will be useful elsewhere. This will be interesting since Pascal's Triangle has had such an effect on other fields of mathematics.

A Variant of Pascal's Triangle

							1							
						1		1						
					2		2		2					
				4		5		5		4				
			8		12		14		12		8			
		16		28		37		37		28		16		
	32		64		94		106		94		64		32	
64		144		232		289		289		232		144		64

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Senior Research Proposal